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Families of automorphic forms

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This is a corrected and slightly expanded version of the notes for my lectures at Córdoba, Argentina, October 24 – November 14, 1995.

The aim was to free the audience from the need to make notes, and to help those readers on their way who want proofs or more details. In the lectures not all subjects mentioned in the notes have been discussed. I give pointers to the literature, but I have made no attempt to be complete, or to point always to original sources.

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1 Modular forms

1.1 Doubly periodic functions

To motivate the study of real analytic modular forms, let us first look at functions on $\mathbf{R}^2$ that are periodic for the lattice $\mathbf{Z}^2$. Fourier theory tells us that $L^2(\mathbf{R}^2 \bmod \mathbf{Z}^2)$ has an orthonormal basis consisting of the functions $(x,y) \mapsto e^{2\pi i(nx+my)}$, with $(n,m) \in \mathbf{Z}^2$. These functions are real analytic, and are eigenfunctions of the Laplace operator $\partial_x^2 + \partial_y^2$. Fourier theory gives the spectral decomposition of (the selfadjoint extension of) the Laplace operator.

We introduce a complex structure by identifying $\mathbf{R}^2$ with $\mathbf{C}$. Then $\mathbf{Z}^2$ corresponds to the lattice $\Lambda = \mathbf{Z} + \mathbf{Z}i$. Among the doubly periodic functions given above only the constant one, with $(n,m) = (0,0)$, is holomorphic. If we want more complex analytic doubly periodic functions, we have to allow poles. Thus one arrives at elliptic functions, which are meromorphic on $\mathbf{C}$ and are invariant under translations by elements of the lattice Λ. See, e.g., [15, chap. 1].

One can stay within the realm of holomorphic functions by relaxing the condition of periodicity. Then one arrives at theta functions, that are holomorphic and satisfy $f(u+\omega) = \varphi_\omega(u) f(u)$, with φ_ω a simple exponential factor. See, e.g., [15, Chap. 18].

We shall consider similar ideas in the situation where $\mathbf{R}^2 \cong \mathbf{C}$ has been replaced by the upper half plane, and the group of translations over elements of $\mathbf{Z}^2 \cong \Lambda$ by a group of non-euclidean motions.

1.2 Upper half plane

We denote the *upper half plane* by $\mathcal{H} := \{ z \in \mathbf{C} : \operatorname{Im} z > 0 \}$. It inherits a complex structure from $\mathbf{C}$.

The group $SL_2(\mathbf{R})$ of real two-by-two matrices with determinant 1 acts on $\mathcal{H}$ by the *fractional linear transformations* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : z \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z := \frac{az+b}{cz+d}$. This is an action of $PSL_2(\mathbf{R}) := SL_2(\mathbf{R})/\{\pm I\}$. Note that the maps $z \mapsto g \cdot z$ are holomorphic.

Any discrete subgroup $\Gamma \subset SL_2(\mathbf{R})$ can play the role of the group of translations over the lattice $\Lambda = \mathbf{Z} + \mathbf{Z}i$ above. In these lectures I restrict myself to the *modular group* $\Gamma_{\rm mod} := SL_2(\mathbf{Z}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{R}) : a,b,c,d \in \mathbf{Z} \}$. So we will be interested in functions $f : \mathcal{H} \to \mathbf{C}$ that satisfy $f\left(\frac{az+b}{cz+d}\right) = f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\rm mod}$.

Hyperbolic geometry. The upper half plane is a model of plane hyperbolic non-euclidean geometry. The hyperbolic lines are the vertical half lines and the half circles with center on the real axis. In a sketch one easily convinces oneself that, given a hyperbolic line ℓ and a point P outside it, there are infinitely many hyperbolic lines through P that do not intersect ℓ. See [20, II.12] for a further discussion. Lehner works mainly on the disk $\{ w \in \mathbf{C} : |w| < 1 \}$, which is equivalent to $\mathcal{H}$ under $w = \frac{z-i}{z+i}$, $z = i\,\frac{1+w}{1-w}$.

The fractional linear transformations by elements of $SL_2(\mathbf{R})$ send hyperbolic lines to hyperbolic lines. In fact, these transformations form the group of all motions of plane hyperbolic geometry.

We shall consider functions on $\mathcal{H}$ that are invariant under the group of non-euclidean motions determined by $\Gamma_{\rm mod}$.

Riemannian metric. The upper half plane carries also a differential-geometrical structure. The metric $(ds)^2 = y^{-2}\left((dx)^2 + (dy)^2\right)$ is invariant under the action of $SL_2(\mathbf{R})$ on $\mathcal{H}$. The length of a smooth curve parametrized by $[a,b] \to \mathcal{H} : t \mapsto (x(t), y(t))$ is $\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, y(t)^{-1}\, dt$. The geodesics for this metric are hyperbolic lines. For a further discussion, see, e.g., [26, 1.4], or [12, 1.1].

This Riemannian metric determines the invariant measure $d\mu(z) := \frac{dx\,dy}{y^2}$ and the hyperbolic Laplacian $y^2\left(\partial_x^2 + \partial_y^2\right)$. See, e.g., [12, 1.6].

Fundamental domain. Let F be the region $\{ z \in \mathcal{H} : |x| \le \frac{1}{2}, |z| \ge 1 \}$. (Here and in the sequel we use the standard convention $x = \operatorname{Re} z$, $y = \operatorname{Im} z$.) The region F is bounded by parts of hyperbolic lines. It has the properties that $\Gamma_{\rm mod} \cdot F = \mathcal{H}$, and that $z_2 = \gamma \cdot z_1$ for $\gamma \in \Gamma_{\rm mod}$ and $z_1, z_2 \in F$, $z_1 \neq z_2$, can only hold if z_1 and z_2 are on the boundary of F; see [36, 1.4]. We know a $\Gamma_{\rm mod}$-invariant function completely if we know its values on F.

The quotient. We can view $Y_{\rm mod} := \Gamma_{\rm mod}\backslash\mathcal{H}$ as the fundamental domain F with boundary identifications: $iy - \frac{1}{2}$ is identified to $iy + \frac{1}{2}$ for $y \ge \frac{1}{2}\sqrt{3}$, and $e^{i\theta}$ is identified to $e^{i(\pi-\theta)}$ for $\theta \in [\pi/3, \pi/2]$. Topologically, $Y_{\rm mod}$ is a sphere with one point deleted. In algebraic geometry one fills up the hole and works with the completion $X_{\rm mod} = Y_{\rm mod} \cup \{\infty\}$, which is the projective line over $\mathbf{C}$. The point ∞ is called the *cusp* of $X_{\rm mod}$.

The Riemannian metric on $\mathcal{H}$ induces a metric on $Y_{\rm mod}$. It is degenerated at the points corresponding to i and to $e^{\pi i/3}$ (the horizon is not 2π, but π, respectively $2\pi/3$ in these points.) The total volume of $Y_{\rm mod}$ is finite, but the tentacle corresponding to the upper part of F is infinitely long.

1.3 Modular forms of weight 0

Invariant functions. Let us consider functions on $\mathcal{H}$ that are invariant under $\Gamma_{\rm mod}$. Some authors call this *modular functions*; I prefer to restrict this name for a more special type of Γ_{mod}-invariant functions.

The first trivial example is the constant function $z \mapsto \sqrt{3/\pi}$. It is of course Γ_{mod}-invariant, holomorphic, and an eigenfunction of the Laplacian. Actually, one often works with minus the Laplacian $L := -y^2\left(\partial_x^2 + \partial_y^2\right)$.

Eisenstein series. Let $s \in \mathbf{C}$. The function $z \mapsto y^s$ is an eigenfunction of L with eigenvalue $s - s^2$, but it is invariant only under the subgroup $\Gamma_{\mathrm{mod}}^{\infty} := \left\{\pm\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right) \in \Gamma_{\mathrm{mod}}\right\}$. A naive approach is to consider the sum

$$\sum_{\gamma \in \Gamma_{\mathrm{mod}}^{\infty}\backslash\Gamma_{\mathrm{mod}}} \mathrm{Im}(\gamma \cdot z)^s.$$

If this converges absolutely, it defines a Γ_{mod}-invariant function. This turns out to work for $\mathrm{Re}\, s > 1$. To check the convergence, note that the sum equals $\frac{1}{2}\sum_{(c,d)}\left(y/|cz+d|^2\right)^s$, where (c,d) runs over the pairs of coprime integers. This defines the Eisenstein series $E(s,z)$, a Γ_{mod}-invariant eigenfunction of L with eigenvalue $s(1-s)$. See [12, 3.1–2] for a more general discussion.

An estimate shows that $E(s,z) = y^s + O\left(y^{1-\mathrm{Re}\, s}\right)$ as $y \to \infty$. So if we go to the cusp ∞, then $|E(s,z)|$ tends to infinity. A computation shows that $E(s)$ is not an element of $L^2(Y_{\mathrm{mod}}, d\mu)$. Here we see an essential difference between the compact quotient $\mathbf{R}^2 \bmod \mathbf{Z}^2$ and $Y_{\mathrm{mod}} = \Gamma_{\mathrm{mod}}\backslash\mathcal{H}$. All continuous doubly periodic functions are bounded, but continuous Γ_{mod}-invariant functions can approach infinity near the cusp.

Poincaré series. $z \mapsto y^s$ is not the only $\Gamma_{\mathrm{mod}}^{\infty}$-invariant eigenfunction of the Laplacian. For suitably chosen functions $h : z \mapsto e^{2\pi i \nu x}p(y)$ the sum $\sum_{\Gamma_{\mathrm{mod}}^{\infty}\backslash\Gamma_{\mathrm{mod}}} h(\gamma \cdot z)$ converges and defines so-called *Poincaré series*, that are Γ_{mod}-invariant eigenfunctions of L with eigenvalue $s - s^2$. The growth at the cusp is in general exponential. These Poincaré series are discussed in [27] and [28]. For a much wider context see [24].

There is another type of Poincaré series, see, e.g., [35, §3]. These functions are elements of $L^2(Y_{\mathrm{mod}}, d\mu)$, but are not eigenfunctions of the Laplacian.

Modular functions. This name we reserve for meromorphic Γ_{mod}-invariant functions on $\mathcal{H}$. By meromorphic we mean that the function should be meromorphic on the whole of X_{mod}. At the cusp this condition amounts to holomorphy and exponential growth in y. The modular functions form the field $\mathbf{C}(j)$, where j is the *modular invariant*, see, e.g., [15, Chap. 3, §3].

1.4 Holomorphic modular forms

Other weights. If we relax the condition of Γ_{mod}-invariance, we can get many nontrivial explicit examples. A *holomorphic modular form* we define to be a function $f : \mathcal{H} \to \mathbf{C}$ that is holomorphic on $\mathcal{H}$, bounded as $y \to \infty$, and satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \text{for all } \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_{\mathrm{mod}}. \tag{1}$$

The number k is called the *weight*.

Eisenstein series. For even $k \geq 4$ the series $\sum_{n,m}'(nz+m)^{-k}$ converges absolutely for each $z \in \mathcal{H}$, and defines a holomorphic function G_k on the upper half plane. The

3.3 The continuous spectrum

Continuation of the Eisenstein series. Inspection of the Fourier expansion of the Eisenstein series on page 7 shows that $E(s,z)$ has a meromorphic continuation as a function of $s \in \mathbb{C}$. The singularities occur at points where $\zeta(2s) = 0$, and at $s = 1$; they all have first order. The transformation behavior under Γ_{mod} stays valid under continuation and taking residues, and so does the relation $L_0 E(z,s) = (s - s^2)\, E(s,z)$. There is also a functional equation, relating $E(1-s,z)$ to $E(s,z)$, which can be figured out from the Fourier expansion and the Maass-Selberg relation. The residue at $s = 1$ is a constant function. The constant functions form a one-dimensional eigenspace of A_0 in H_0^c.

The proof of the meromorphic continuation of the Eisenstein series from inspection of the Fourier series can be found in [22, IV.3]. The approach in [16, Ch. XIII] is based on the summation formula of Poisson. Both proofs are bound to an arithmetical situation. For discrete groups Γ not related to the modular group, they do not work. There one needs the resolvent of A_0. Several approaches based on this idea of Selberg one finds explained in [11]. The resolvent is also used in the approaches of Faddeev, [9], and Lax and Phillips, [19]. Faddeev's proof is expanded in [16, Ch. XIV]. The approach of Colin de Verdière, [7], looks more like that of Lax and Phillips; I discuss it in Subsection 3.4.

Theorem 2 *Let H_0^c be the orthogonal complement in H_0^c of the space $\mathbb{C}\cdot 1$ of constant functions. The Hilbert spaces H_0^c and $L^2\left((0,\infty), \frac{dt}{2\pi}\right)$ are isomorphic. The isomorphism between these spaces is induced by the* Eisenstein transform

$$\mathcal{E}f(t) = \int_F E(it + 1/2, z) f(z)\, d\mu(z)$$

for bounded $f \in H_0$. The restriction of A_0 to H_0^c corresponds to multiplication by $\frac{1}{4} + t^2$ under this isomorphism.

This result is discussed in [12, Ch. 7], and in [16, Ch.XIV]. A more general point of view one finds in [16, Ch. XIII].

For bounded f in H_0 the integral defining $\mathcal{E}f$ converges, and defines a continuous square integrable function on $(0,\infty)$. This map $f \mapsto \mathcal{E}f$ is continuous in L^2-sense. Hence it extends as a map $H_0 \to L^2\left((0,\infty), \frac{dt}{2\pi}\right)$. The theorem states that the kernel of this extension is $H_0^0 \oplus \mathbb{C}\cdot 1$, and that the induced map $H_0^c \to L^2\left((0,\infty), \frac{dt}{2\pi}\right)$ is a unitary map.

Spectral decomposition. We have seen that $H_0 = (H_0^0 \oplus \mathbb{C}\cdot 1) \oplus H_0^c$. The restriction to $H_0^0 \oplus \mathbb{C}\cdot 1$ of self-adjoint operator A_0 has a discrete spectrum; the restriction to H_0^c has a continuous spectrum. For $r \in (-12, 0) \cup (0, 12)$ the operator A_r has a discrete spectrum in H_r.

3.4 Pseudo-Laplacian

In the spectral theory of modular forms the presence of the cusp ∞ causes problems. In most proofs of the meromorphic continuation of the Eisenstein series, these difficulties are overcome by giving the kernel of integral operators a special treatment near the cusp. Interesting in the approach of Colin de Verdière, [7], is the use of another operator, the *pseudo-Laplacian*.

I explain the main ideas in the situation of $r = 0$. I write $H = H_0$. The ideas are from [7]; I refer to [5] for proofs.

Self-adjoint operator and sesquilinear form. For smooth $f, g \in H$ with compact support in Y_{mod} we have

$$(L_0 f, g) = \int_F \nabla f \cdot \overline{\nabla g}\, dx\, dy,$$

where $\nabla f \cdot \overline{\nabla g}$ is the scalar product of the gradients; see [12, 4.1]. We define the subspace D of those $f \in H$ for which $y\|\nabla f\|$ is square integrable on F. We give D the structure of a Hilbert space by defining

$$(f, g)_D := (f, g) + \int_F \nabla f \cdot \overline{\nabla g}\, dx\, dy.$$

The inclusion $D \to H$ is not a compact operator, due to the presence of the cusp. (See, e.g., [5, 6.2, 6.4].)

On the space D we have the sesquilinear form $\mathfrak{s}[f, g] = \int_F \nabla f \cdot \overline{\nabla g}\, dx\, dy$, extending the linear form $(L_0 f, g)$ on smooth compactly supported elements. Now we use the theorem that sesquilinear forms of this type correspond to self-adjoint operators (see, e.g., [13, Chap. VI, §2.1]). This implies that there is a unique self-adjoint operator A in H with domain contained in D, such that $\mathfrak{s}[f, g] = (Af, g)$ for all $f \in \mathrm{dom}(A)$ and $g \in D$. The choice of $\mathfrak{s}$ implies that this operator is the self-adjoint extension A_0 of the Laplace operator. (See, e.g., [5, 6.5].) One calls this way of obtaining A_0 the Friedrichs extension.

Truncation. The resolvent of the operator A is not compact. We go over to smaller spaces to arrive at another operator, that has a compact resolvent.

We fix $a > 5$. The map $f \mapsto F_0 f$ has an L^2-interpretation. We define ${}^aH$ as the subspace of those $f \in H$ for which $F_0 f = 0$ on (a, ∞). So ${}^aH$ is a closed subspace of H. We define ${}^aD = {}^aH \cap D$, with the scalar product $(\cdot, \cdot)_D$. This inclusion ${}^aD \to {}^aH$ is compact, [5, 8.4.8]. The sesquilinear form $\mathfrak{s}$ restricted to ${}^aD$ corresponds to a self-adjoint operator ${}^aA$ in ${}^aH$, with compact resolvent. (See [5, 9.2.2], with $\varphi = 0$.)

The relation between this operator ${}^aA$ and the Laplacian is discussed in [5, 9.2.5]: Consider $f \in \mathrm{dom}({}^aA)$, and test it against a smooth $g \in H$ with compact support in Y_{mod}:

$$({}^aAf, g) = (f, L_0 g) + d(f)\overline{F_0 g(a)}, \tag{3}$$

for some $d(f) \in \mathbb{C}$. If $F_0 f$ is a smooth function on $(5, a]$, then

$$d(f) = \lim_{y \uparrow a} -(F_0 f)'(y).$$

We shall show the proof of this relation in 3.4.1.

These results mean that, in distribution sense, ${}^aA$ is equal to the Laplacian L_0 everywhere on F, except at the level $y = a$, where it differs from L_0 by a delta function acting on F_0. The operator ${}^aA$ is the *pseudo-Laplacian* of Colin de Verdière.

Continuation. We want to show that there is a meromorphic family $s \mapsto E(s)$ of modular forms with polynomial growth and eigenvalue $s \mapsto s - s^2$, that coincides with the Eisenstein series for $\mathrm{Re}\, s > 1$. In order to do this, we solve $(L_0 - s + s^2)\, X = 0$.

Take a smooth function $\varphi_s : (0,\infty) \to \mathbf{C}$ equal to 0 on $(0,5)$ and on $(\frac{5+a}{2},\infty)$ equal to a linear combination of y^s and y^{1-s} such that $\varphi_s(a) = 1$. We define $h_s : F \to \mathbf{C}$ by $h_s(z) = \varphi_s(y)$, and extend it to $\mathcal{H}$ in a Γ_{mod}-invariant way. We define $k_s = (L_0 - s + s^2)\, h_s$. This yields a holomorphic family $\mathbf{C} \to {}^aH : s \mapsto k_s$.

Let $R(s) = ({}^aA - s + s^2)^{-1}$ be the resolvent. We use this meromorphic family of operators in ${}^aH$ to define a meromorphic family $s \mapsto g_s$ of elements of ${}^aH$ by $g_s = R(s)k_s$. The Fourier term $F_0 g_s$ is given by a linear combination $c_1(s)y^s + c_2(s)y^{1-s}$ on $(\frac{5+a}{2}, a)$, and vanishes above $y = a$. Define $\tilde{g}_s$ on F by $\tilde{g}_s(z) = g_s(z)$ for $y \leq a$ and $\tilde{g}_s(z) = g_s(z) + c_1(s)y^s + c_2(s)y^{1-s}$ for $y \geq a$. This defines a Γ_{mod}-invariant function $\tilde{g}_s$.

We combine the two solutions h_s and $\tilde{g}_s$ of $(L_0 - s + s^2)\, X = k_s$ to the solution $\tilde{E}_s := h_s - \tilde{g}_s$ of $(L_0 - s + s^2)\, X = 0$. The condition $F_0\tilde{E}_s(a) = 1$ determines this solution uniquely. Indeed, the non-zero difference of two solutions would be a meromorphic family of eigenfunctions of ${}^aA$ with eigenvalue ranging through $\mathbf{C}$, which is impossible for a self-adjoint operator.

The square integrability of g_s implies that its Fourier terms of non-zero order are quickly decreasing, so the same holds for $\tilde{E}_s$. The Fourier term of order 0 looks like $F_0\tilde{E}_s(y) = A(s)y^s + B(s)y^{1-s}$. If A would vanish identically, the family would be square integrable for all s with $\operatorname{Re} s > \frac{1}{2}$, which is impossible. Thus $s \mapsto \frac{1}{A(s)}\tilde{E}_s$ is a meromorphic family of modular forms with a Fourier series of the same type as that of $E(s)$. The Maass-Selberg relation (see page 7) implies that it is the Eisenstein series for $\operatorname{Re} s > 1$.

It is much more complicated to prove the meromorphic continuation of the Eisenstein series in this way than by looking at the Fourier expansion. But this proof works in a much more general situation, and can be extended to give the continuation in weight r and spectral parameter s jointly.

3.4.1 Proof of (3)

Here I give the proof of formula (3), and the expression for $d(f)$ if $F_0 f$ is smooth on $(5, a]$. This gives me the opportunity to go deeper into the gradient ∇f.

For $f \in C^\infty(\Gamma_{\text{mod}}\backslash\mathcal{H})$ we have $\nabla f = \binom{f_x}{f_y}$. The functions $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ have a complicated transformation behavior under the modular group. It is better to work with $\mathbf{E}_0^+ f = 2iyf_x + 2yf_y$ and $\mathbf{E}_0^- f = -2iyf_x + 2yf_y$. These functions $\mathbf{E}^\pm f$ have the transformation behavior of weight ± 2 for the trivial multiplier system. So we can define D_0 as the subspace of those $f \in H_0$ for which the distribution derivatives $\mathbf{E}_0^\pm f$ are given by square integrable functions.

The distribution derivative $\mathbf{E}_0^\pm f$ is determined by

$$\left(\mathbf{E}_0^\pm f, k_\pm\right) = -\left(f, \mathbf{E}_{\pm 2}^\mp k_\pm\right) \text{ for all smooth } k_\pm \in K_{\pm 2}.$$

Here K_{2k} means the space of smooth functions h on $\mathcal{H}$ that satisfy $h(\gamma \cdot z) = e^{2ik\arg(c_\gamma z + d_\gamma)}h(z)$ for all $\gamma \in \Gamma_{\text{mod}}$, for which $|h|$ has compact support in Y_{mod}. The differential operator $\mathbf{E}_{2k}^\pm$ is defined as $\pm 2iy\partial_x + 2y\partial_y \pm 2k$. (It is the operator in weight $2k$ coming from the element $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \pm i\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ of the complexified Lie algebra.) If f is a smooth function, this relation between $\mathbf{E}_0^\pm$ and $\mathbf{E}_{\pm 2}^\mp$ can be checked by applying Stokes' theorem to the differential form $\pm\frac{2i}{y} f\bar{h} - \frac{2}{y} f\bar{h}$ on a truncated fundamental domain (take the truncation above the support of $|h|$).

Note that $(y\nabla f, y\nabla h) = \frac{1}{8}\sum_{\pm}\left(\mathbf{E}_0^{\pm} f, \mathbf{E}_0^{\pm} h\right)$.

Formula (3). Let $g \in K_0 = C_c^{\infty}(\Gamma_{\mathrm{mod}}\backslash\mathcal{H})$. So g is smooth, Γ_{mod}-invariant, and has compact support in Y_{mod}. First we consider the case that $F_0 g = 0$ on (b, ∞) for some $b \in (5, a)$. Then $g \in {}^aD$, and $\mathbf{s}[f, g] = \frac{1}{8}\sum_{\pm}\left(\mathbf{E}_0^{\pm} f, \mathbf{E}_0^{\pm} k\right) = -\frac{1}{8}\sum\left(f, \mathbf{E}_{\pm 2}^{\mp}\mathbf{E}_0^{\pm} g\right) = (f, L_0 g)$. On the other hand, $\mathbf{s}[f, g] = ({}^aAf, g)$, so for this g formula (3) holds.

This leaves only one type of functions g still to be considered: $g(z) = g_\varphi(z) = \varphi(y)$ for $z \in F$, $\varphi \in C_c^{\infty}(0, \infty)$, $\mathrm{Supp}\,\varphi \subset (5, \infty)$. If the support of φ is contained in $(5, a)$, then g_φ is of the type already considered. If the support of φ is contained in (a, ∞), then all terms in (3) vanish (remember that f and ${}^aAf$ are elements of ${}^aA$). Let us consider the antilinear form

$$\delta : \varphi \mapsto ({}^aAf, g_\varphi) - (f, L_0 g_\varphi)\,.$$

It is given by integrating φ and its derivatives up to second order against square integrable functions; hence δ is a distribution in the second Sobolev space on $(5, \infty)$. The support of δ is contained in $\{a\}$. Theorem 4 in [18, XI, §4] shows that δ is a linear combination of the delta-distribution at a and its derivatives. If we show that δ is in the first Sobolev space (integration of φ and φ' against square integrable functions), then only the derivative of order zero can occur.

We have $({}^aAf, g_\varphi) = \int_5^a F_0{}^aAf(y)\overline{\varphi(y)}\frac{dy}{y^2}$. This is continuous for the first Sobolev norm. Furthermore

$$(f, L_0 g_\varphi) = -\frac{1}{8}\sum_{\pm}\left(\mathbf{E}_0^{\pm} f, \mathbf{E}_0^{\pm} g_\varphi\right) = -\frac{1}{8}\sum_{\pm}\int_5^{\infty} F_0\mathbf{E}_0^{\pm} f\,\overline{2y\varphi'(y)}\,\frac{dy}{y^2},$$

which is also continuous for that norm. So the proof of (3) is finished.

Explicit form of $d(f)$. Let $f \in \mathrm{dom}({}^aA) \subset {}^aD$, and suppose that $F_0 f$ is given by a smooth function on $(5, a]$. So there is a smooth function Φ on $(5, a]$ such that

$$y \mapsto \begin{cases} \Phi(y) & \text{if } y \le a \\ 0 & \text{if } y > a \end{cases}$$

represents $F_0 f$ on $(5, \infty)$. (At $y = a$ we view the derivatives of Φ as left derivatives.) We want to show that $d(f) = -\Phi'(a)$.

First we show that $\Phi(a) = 0$, and that $F_0\mathbf{E}_0^{\pm} f$ is represented by $y \mapsto 2y\Phi'(y)$ on $(5, a)$. (Of course, $F_0\mathbf{E}_0^{\pm} f$ is represented by 0 on (a, ∞); so this determines $F_0\mathbf{E}_0^{\pm} f$ on $(5, \infty)$ in L^2-sense.)

Let $\varphi \in C_c^{\infty}(0, \infty)$, $\mathrm{Supp}\,\varphi \subset (5, \infty)$. Define $k \in K_{\pm 2}$ by $k(z) = \varphi(y)$ for $z \in F$. We have

$$\begin{aligned}
\left(\mathbf{E}_0^{\pm} f, k\right) &= -\left(f, \mathbf{E}_{\pm 2}^{\mp} k\right) = -\int_5^a \Phi(y)\overline{(2y\varphi'(y) - 2\varphi(y))}\frac{dy}{y^2} \\
&= \int_5^a 2y\Phi'(y)\overline{\varphi(y)}\frac{dy}{y^2} - \frac{2}{a}\Phi(a)\overline{\varphi(a)}.
\end{aligned}$$

Application of this result with $\mathrm{Supp}(\varphi) \subset (5, a)$ shows that $y \mapsto 2y\Phi'(y)$ represents $F_0\mathbf{E}_0^{\pm} f$ on $(5, a)$. But then we know for all φ with $\mathrm{Supp}\,\varphi \subset (5, \infty)$ that

$$\left(\mathbf{E}_0^{\pm} f, k\right) = \int_5^a 2y\Phi'(y)\overline{\varphi(y)}\frac{dy}{y^2}.$$

So the term with $\Phi(a)$ should vanish for all such φ, hence $\Phi(a) = 0$. Note that we have used $f \in {}^aD$, but not that $f \in \operatorname{dom} {}^aA$.

Now we turn to $\delta(\varphi) = d(f)\overline{\varphi(a)} = ({}^aAf, g) - (f, L_0 g)$, with $g = g_\varphi \in K_0$ as above. We have

$$\begin{aligned}
({}^aAf, g) &= \int_5^a F_0({}^aAf)(y)\,\overline{\varphi(y)}\,\frac{dy}{y^2} \\
(f, L_0 g) &= -\int_5^a \Phi(y)\overline{\varphi''(y)}\,dy \\
&= \int_5^a \Phi'(y)\overline{\varphi'(y)}\,dy \qquad\qquad (\text{use } \Phi(a) = 0) \\
&= -\int_5^a \Phi''(y)\overline{\varphi(y)}\,dy + \Phi'(a)\overline{\varphi(a)} \\
\delta(\varphi) &= \int_5^a \left(F_0({}^aAf)(y) + y^2\Phi''(y)\right)\overline{\varphi(y)}\,\frac{dy}{y^2} - \Phi'(a)\,\overline{\varphi(a)}.
\end{aligned}$$

As δ has support in $\{a\}$, we conclude that $F_0{}^aAf$ is represented by $y \mapsto -y^2\Phi''(y)$ on $(5, a)$, and that $d(f) = -\Phi'(a)$.

4 Perturbation

Now we turn to the question how the spectrum of A_r varies as r runs through $(-12, 12)$.

4.1 Transformation

To apply the perturbation theory of linear operators discussed in Chapter VII of [13], we need to have a family of operators acting in a fixed space.

This can be arranged by multiplication by the powers of the eta function of Dedekind. Remember that $\eta_r(z) = y^{r/2}\eta(z)^{2r}$. The map $f \mapsto \eta_{-r}f$ sends functions with transformation behavior described by v_r to Γ_{mod}-invariant functions. But this transformation has several drawbacks. Firstly, the map is not unitary. So it is better to use $f \mapsto e^{-2ir\,\mathrm{Im}\log\eta}f$. In this way we get a unitary isomorphism $H_r \to H_0$. Instead of considering L_r and its self-adjoint extension A_r in the spaces H_r, we can consider the family $r \mapsto e^{-2ir\,\mathrm{Im}\log\eta} \circ L_r \circ e^{2ir\,\mathrm{Im}\log\eta}$ in the fixed space $H = H_0$.

For z near the cusp ∞ the factor $e^{2ir\,\mathrm{Im}\log\eta(z)}$ is complicated. The Fourier terms of the function of weight r do not correspond term by term to those in the Fourier expansion of the Γ_{mod}-invariant function. To achieve that, we replace $2\,\mathrm{Im}\log\eta$ by a smooth real valued function t on $\mathcal{H}$ with the properties

i) $t(z) = \frac{1}{6}\pi x$ for all $y \geq 5$.

ii) $t(\gamma\cdot z) = \alpha(\gamma) + \arg(cz+d) + t(z)$ for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathrm{mod}}$. Here $\alpha : \Gamma_{\mathrm{mod}} \to \frac{\pi}{6}\mathbb{Z}$ is the function such that $v_r(\gamma) = e^{ir\alpha(\gamma)}$ for all $\gamma \in \Gamma_{\mathrm{mod}}$ and all $r \in \mathbb{C}$.

This can be arranged in such a way that $t(z) = 2\,\mathrm{Im}\log\eta(z)$ for $z \in F$, $y \leq 3$ (Lemma 3.2 in[1]). Above the level $y = 5$, the Fourier terms in weight r correspond to the Fourier terms in weight 0.

Once we have chosen the transformation function t, we can work completely in $H = H_0$. Let $L(r) = e^{-irt} \circ L_r \circ e^{irt}$. It is an exercise to see that $L(r) = L_0 + rL^{(1)} + r^2 L^{(2)}$, with well determined differential operators $L^{(1)}$ and $L^{(2)}$.

4.2 Holomorphic family of operators

For $f, g \in C_c^\infty(\Gamma_{\mathrm{mod}} \backslash \mathcal{H})$ we can write

$$(L(r)f, g) = \mathrm{s}[f,g] + r\mathrm{s}^{(1)}[f,g] + r^2 \mathrm{s}^{(2)}[f,g],$$

with the form s that we discussed on page 10, and two other sesquilinear forms $\mathrm{s}^{(1)}$ and $\mathrm{s}^{(2)}$. Some estimates show that $\mathrm{s}^{(1)}$ and $\mathrm{s}^{(2)}$ are defined on ${}^aD$ (proof in [5, Lemma 8.4.11]). The expression $\mathrm{s}(r) := \mathrm{s} + r\mathrm{s}^{(1)} + r^2\mathrm{s}^{(2)}$ makes sense for $r \in \mathbf{C}$, not only for $r \in (-12, 12)$. More work, see [5, §9.1], shows that there is a neighborhood V_0 of 0 in $\mathbf{C}$ on which $r \mapsto \mathrm{s}(r)$ is a *holomorphic family of type (a)* in the sense of [13, Chap. VII, §4.2]. I do not state what that means, but list consequences for the present situation: For each $r \in V_0$ there is a densely defined closed operator ${}^aA(r)$ in ${}^aH$ such that $\mathrm{s}(r)[f,g] = ({}^aA(r)f, g)$ for $f \in \mathrm{dom}({}^aA(r))$ and $g \in {}^aD$. This operator extends $L(r)$. For $r = 0$ it is the pseudo-Laplacian. All ${}^aA(r)$ have a compact resolvent. For $r \in V_0$ such that $\bar{r} \in V_0$ we have ${}^aA(\bar{r}) = {}^aA(r)^*$; in particular, ${}^aA(r)$ is self-adjoint for $r \in V_0 \cap \mathbf{R}$. For real $r \in V_0$ the spectrum of ${}^aA(r)$ is discrete, the eigenvalues are holomorphic in r on a neighborhood of $V_0 \cap \mathbf{R}$; this neighborhood may depend on the eigenvalue. If $\lambda : V_0 \cap \mathbf{R} \to \mathbf{R}$ is an eigenvalue, then there is a finite number of eigenfamilies $r \mapsto \varphi_j(r)$, each extending holomorphically to a neighborhood of $V_0 \cap \mathbf{R}$, such that the $\varphi_j(r)$ form an orthonormal basis of the eigenspace $\ker({}^aA(r) - \lambda(r))$ for each $r \in V_0 \cap \mathbf{R}$. To summarize: The family $r \mapsto {}^aA(r)$ is as well behaved as one can reasonably wish it to be. (These facts are based on the following results in [13]: Chap. VI, Theorem 2.1; Chap. VII, Theorem 4.8, Theorem 4.3, Theorem 3.9, and Remark 4.22.)

After a gluing procedure as indicated in [5, 9.2.8] we can assume that V_0 is a neighborhood of $(-12, 12)$ in $\mathbf{C}$, and $V_0 \cap \mathbf{R} = (-12, 12)$.

What does this mean for families of cusp forms? Let $\lambda : (-12, 12) \to \mathbf{R}$ be an eigenvalue of the family $r \mapsto {}^aA(r)$, and let $\varphi : (-12, 12) \to {}^aH$ be a corresponding eigenfamily. For $y = \mathrm{Im}\, z \in (5, a)$, the Fourier expansion has the form:

$$\begin{aligned}
\varphi(r; z) &= b_0(r)\mu(r, s(r); y) + c_0(r)\mu(r, 1 - s(r); y) \\
&\qquad + \sum_{\nu \neq 0} c_\nu(r) e^{2\pi i \nu x} \omega^{\nu + r/12}(r, s(r); y), \\
\omega^n(r, s; y) &= W_{r\,\mathrm{sign}(\mathrm{Re}\, n)/2,\, s-1/2}(4\pi n y\, \mathrm{sign}(\mathrm{Re}\, n)), \\
\mu(r, s; y) &= y^s e^{-\pi r y/6} {}_1F_1\left[\begin{matrix} s - r/2 \\ 2s \end{matrix} \middle| \frac{\pi r y}{3}\right] \qquad (4) \\
&= y^s e^{-\pi r y/6} \sum_{n=0}^{\infty} \frac{\left(s - \frac{r}{2}\right)_n}{(2s)_n n!} \left(\frac{\pi r y}{3}\right)^n,
\end{aligned}$$

where $(a)_0 = 1$
and $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$.

(For convenience I assume that $\lambda(r) = s(r) - s(r)^2$, with $s : (-12, 12) \to \mathbf{C}$ real analytic.) The term of order zero vanishes above $y = a$, the other terms are given by the same formula. Furthermore, from $\varphi(r) \in {}^aD$ we can show that $b_0(r)\mu(r, s(r); a) + c_0(r)\mu(r, 1 - s(r); a) = 0$ in the same way as we proved $\Phi(a) = 0$ in 3.4.1. All coefficients $b_0(r)$ and $c_\nu(r)$ are real analytic in r.

Define $\tilde{\varphi}(r)$ as the Γ_{mod}-invariant function given by $\tilde{\varphi}(r; z) = \varphi(r; z)$ if $z \in F$, $y \leq a$, and by the Fourier series given above if $z \in F$, $y \geq a$. Then $r \mapsto e^{irt}\tilde{\varphi}(r)$ is a family of modular forms of varying weight, with in general exponential growth. Only if b_0 and c_0 are the zero functions, the resulting family is a family of cusp forms. The value at $r = 0$ is a cuspidal Maass form of weight zero. I do not know whether families with $b_0 = c_0 = 0$ really exist.

Other truncation schemes. In the construction of the family ${}^aA$ we have given a special treatment to the Fourier term with $e^{\pi i r x/6}$. If we truncate more Fourier terms, then we obtain a family of operators on a larger interval. For example, truncation of all Fourier terms with $|\nu| \leq 5$ gives a holomorphic family of operators on a neighborhood of $(-72, 72)$. This leads to eigenvalues and associated families of eigenfunctions on a neighborhood of $(-72, 72)$ as well.

We can also apply the reasoning sketched above without any truncation at all. Then $r = 0$ is a dangerous point. We obtain the existence of a family A of operators on neighborhoods of $(0, 12)$, and of $(-12, 0)$. For $r \in (-12, 12)$, $r \neq 0$, the operator $A(r)$ corresponds to the self-adjoint extension A_r of L_r (see page 8). This means that the eigenvalues and basis vectors of H_r $(r \neq 0)$ given by Theorem 1 occur in holomorphic families on neighborhoods of $(0, 12)$, and of $(-12, 0)$.

Further results. With a more complicated analysis, [3] gives the following results. In stating the results I restrict myself to $(0, 12)$, and use the fact that real analyticity on $(0, 12)$ is the same as holomorphy on a neighborhood of $(0, 12)$.

Theorem 3 *There are a countable set Λ of real analytic functions $(0, 12) \to \mathbf{R}$ and for each $\lambda \in \Lambda$ a real analytic function $\psi_\lambda : (0, 12) \times \mathcal{H} \to \mathbf{C}$ such that $z \mapsto \psi_\lambda(r, z)$ is a modular cusp form of weight r, with multiplier system v_r and eigenvalue $\lambda(r)$ for each $r \in (0, 12)$.*

$\{\psi_\lambda(r) : \lambda \in \Lambda\}$ is a complete orthonormal system in H_r for each $r \in (0, 12)$.

$\lambda_\eta : r \mapsto \frac{r}{2} - \frac{r^2}{4}$ is an element of Λ and $\psi_{\lambda_\eta}(r)$ is a multiple of η_r. For all other elements $\lambda \in \Lambda$, we have $\lambda > \frac{1}{4}$ on $(0, 12)$.

For $\lambda \in \Lambda$, $\lambda \neq \lambda_\eta$, exactly one of the following statements holds:

i) λ and ψ_λ have no analytic extension across $r = 0$, and $\lim_{r \downarrow 0} \lambda(r) = \frac{1}{4}$. The Fourier term $F_{r/12}\psi_\lambda(r)$ is non-zero for some r.

ii) λ and ψ_λ have an analytic extension to $(-12, 12)$. For each $r \in (-12, 12)$ the value ψ_λ is a modular cusp form of weight r, with multiplier system v_r and eigenvalue $\lambda(r)$. The Fourier term $F_{r/12}\psi_\lambda(r)$ is zero for all r.

The elements of Λ are all different. The theorem implies that generically all spaces of modular cusp forms that are not related to holomorphic modular forms have dimension 1. Spaces with higher dimension may occur at points where the graphs of elements of Λ intersect each other. I do not know whether that really happens.

Those $\lambda \in \Lambda$ and ψ_λ that come under case ii) of the theorem are precisely the eigenvalues and eigenfamilies that the families of operators A and ${}^aA$ have in common.

4.3 Continuation of the Eisenstein series

Let us return to the family $r \mapsto {}^aA(r)$ on a neighborhood of $(-12, 12)$, discussed in Subsection 4.2. It has been obtained by truncation of one Fourier term. It is a *holomorphic family of type (B)* in the sense of [13, VII, §4.2]. That is a quite complicated concept of holomorphy.

A more simple concept is a *bounded holomorphic family of operators*. Let $\Omega \subset \mathbf{C}^k$ be an open set. A map B from Ω to the bounded operators in a Hilbert space is a *bounded-holomorphic family* if for each $(v_1, \ldots, v_k) \in \Omega$ there is a neighborhood $\Omega_v \subset \Omega$ of $(v_1, \ldots, v_k)$ such that for each $(w_1, \ldots, w_k)$ we have a power series expansion

$$B(w_1, \ldots, w_k) = \sum_{n_1 \geq 0, \ldots, n_k \geq 0} (w_1 - v_1)^{n_1} \cdots (w_k - v_k)^{n_k} C_{n_1, \ldots, n_k}$$

with bounded operators $C_{n_1,\ldots,n_k}$, converging absolutely in operator norm. A family of operators is bounded-meromorphic if on a neighborhood of each point it can be written in the form $\frac{1}{\psi}B$, with B a bounded-holomorphic family of operators and ψ a holomorphic function that is not identically zero.

A nice property of the family ${}^aA$ is that its resolvent

$$R(r, s) = \left({}^aA(r) - s + s^2\right)^{-1}$$

is a bounded-meromorphic family in (r, s). With some work, that I do not show here, we can extend to this situation the method of Colin de Verdiére discussed in Subsection 3.4. We work in $H = H_0$, and transform $R(r, s)$ to a meromorphic family of bounded operators in this space. We replace φ_s on page 11 by a suitable linear combination of $\mu(r, s)$ and $\mu(r, 1-s)$ (see (4)), and follow the reasoning in Subsection 3.4. We obtain a meromorphic family in H_0 that corresponds to a family with varying weight (after multiplication by e^{irt}). The reasoning is carried out in [5, §9.4, §10.2] in a more general context. In this way we obtain the following continuation of the Eisenstein series in two variables:

Theorem 4 *There are a neighborhood V_0 of $(-12, 12)$ in $\mathbf{C}$ and a meromorphic family E of modular forms on $V_0 \times \mathbf{C}$. The Fourier expansion is*

$$\begin{aligned} E(r, s; z) &= e^{\pi i r x/6}\left(\mu(r, s; y) + C_0(r, s)\mu(r, 1-s; y)\right) \\ &\quad + \sum_{\nu \neq 0} C_\nu(r, s) e^{2\pi i(\nu + r/12)x} \omega^{\nu + r/12}(r, s; y), \end{aligned}$$

with the C_ν meromorphic on $V_0 \times \mathbf{C}$. These conditions determine E uniquely.

There are functional equations $C_0(r, s)C_0(r, 1-s) = 1$, $E(r, 1-s) = C_0(r, 1-s)E(r, s)$, and $E(r, s; -\bar{z}) = E(-r, s; z)$.

The family $s \mapsto E(0, s)$ exists as a meromorphic family of modular forms of weight 0. On $\mathrm{Re}\, s > 1$ it is given by the Eisenstein series in weight 0.

By a holomorphic family of modular forms f on $V_0 \times \mathbf{C}$ I mean an real analytic function f on $V_0 \times \mathbf{C} \times \mathcal{H}$ that is holomorphic in the first two variables, such that $z \mapsto f(r, s, z)$ is a modular form of weight r, multiplier system v_r and eigenvalue $s - s^2$. (The analyticity in (r, s, z) and holomorphy in (r, s) means that locally there are power series expansions in $r - r_0$, $s - s_0$, $z - z_0$ and $\overline{z - z_0}$. A family f of modular forms is meromorphic if it has locally a representation of the form $f = \frac{1}{\psi}h$, with h a holomorphic family of modular forms, and $(r,s) \mapsto \psi(r,s)$ a non-zero holomorphic function.

The existence of the restriction $s \mapsto E(0,s)$ to the complex line $r = 0$ is not trivial; this line could have been contained in the set of singularities.

Singularities on the critical line. The Fourier expansion on page 7 shows that $s \mapsto E(0,s)$ is holomorphic at all points of the line $\mathrm{Re}\, s = \frac{1}{2}$. This does not mean that E is holomorphic at the points $(0,s)$ with $\mathrm{Re}\, s = \frac{1}{2}$. In fact, singularities at these points are related to the possibility of extending cuspidal eigenvalues, [3, Theorem 2.19]:

Theorem 5 *Let $t \in \mathbf{R}$. Denote by $\mathcal{M}(t)$ the space of cuspidal Maass forms of weight 0 with eigenvalue $\frac{1}{4} + t^2$, and by $\mathcal{M}_e(t)$ the subspace of $\mathcal{M}(t)$ spanned by the values $\psi_\lambda(0)$ where λ runs through the $\lambda \in \Lambda$ satisfying ii) in Theorem 3 and $\lambda(0) = \frac{1}{4} + t^2$. The following statements are equivalent:*

i) $\mathcal{M}_e(t) \neq \mathcal{M}(t)$.

ii) The family E is not holomorphic at $(0, \frac{1}{2} + it)$.

I do not know whether the equivalent statements i) and ii) are true for all, some or no values of t for which $\mathcal{M}(t) \neq \{0\}$.

Poincaré series with exponential growth. On page 3 I have mentioned quickly growing Poincaré series. Let $h(z) = e^{\pi i r x/6}\mu(r,s;y)$. For $\mathrm{Re}\, s > 1$, $r \in (0,12)$, the series

$$\sum_{\gamma \in \Gamma_{\mathrm{mod}}^{\infty} \backslash \Gamma_{\mathrm{mod}}} v_r(\gamma)^{-1} e^{-ir \arg(c_\gamma z + d_\gamma)} h(\gamma \cdot z)$$

converges absolutely, and defines the Poincaré series $P_r(s;z)$, which is a modular form of weight r, multiplier system v_r and eigenvalue $s - s^2$. This is the Poincaré series associated to the term with $e^{\pi i r x/6}$ in the Fourier expansion. One can build similar exponentially growing Poincaré series for all Fourier terms.

The Maass-Selberg relation enables us to relate these Poincaré series to the family E. Let $v(r,s) := \left(\frac{\pi r}{3}\right)^s \Gamma(1-2s)/\,\Gamma(1-s-r/2)$. Define for $r \in V_0$, $\mathrm{Re}\, r > 0$:

$$P(r,s) := \frac{v(r,1-s)}{v(r,1-s) - v(r,s)C_0(r,s)} E(r,s).$$

This gives a meromorphic family P, such that $P(r,s) = P_r(s)$ for $r \in (0,12)$, $\mathrm{Re}\, s > 1$. (See [5, §10.3].)

Singularities at integral points. The Eisenstein series of weight 0 is holomorphic on $\mathrm{Re}\, s > 1$, but the continuation E in two variables may have singularities at the points $(0,s)$ with $\mathrm{Re}\, s > 1$.

In [5, §12.3] one sees, in a much wider context than the present one, that singularities of families like $(r,s) \mapsto E(r,s)$ at points (r,s) with $r \in \mathbf{R}$ can arise in two ways. The first is the most essential one. If for (r,s) there are cusp forms with weight r and eigenvalue $s - s^2$, the resolvent will have a singularity at this point. This singularity often propagates through the construction of E.

The other cause is more trivial. The family E has been characterized by the special form of its Fourier term of order $\frac{r}{12}$. When we express the Fourier term in the basis $\mu(r,s)$, $\mu(r,1-s)$, the first coefficient should be 1. This is not a sensible condition at points where $\mu(r,s)$, $\mu(r,1-s)$ do not form a basis. So we may expect singularities at $(r,s) = (0,\frac{l}{2})$, $l \in \mathbb{Z}$. We can analyze these singularities by using a more suitable basis of the space of Fourier terms. [5, Proposition 12.4.2, part iii)] gives a general result. For the family E one can prove that for $l \in \mathbb{N}$, $l \geq 3$, the family E^l given below is holomorphic on a neighborhood of $(0,l/2)$ with the Eisenstein series $E(l/2;z)$ as its value at $(0,l/2)$.

$$E^l(r,s) = \frac{1}{1 - w_l(r,s)C_0(r,s)}E(r,s),$$
$$w_l(r,s) = \left(\frac{\pi r}{3}\right)^{l-1}\frac{\Gamma(1-2s)}{\Gamma(2s-1)}\left(1-s-\frac{r}{2}\right)_{l-1}.$$

Note that the meromorphic function $(r,s) \mapsto w_l(r,s)$ has restriction 0 to the line $r = 0$.

4.4 Distribution results

The singularities of the family E at points $(0,l/2)$ at first seem to form an annoying property of the continuation of the Eisenstein series in two variables. But it turns out that they can be used to get distribution results for various quantities. I give a short indication how to obtain the distribution results for Dedekind sums in [4]. There are many explicit formulas; I give only a few of these. See [6] for a generalized version, with more side remarks. Matthes has obtained distribution results related to Fourier expansions along periodic geodesics, see [23].

Fourier coefficients. The restrictions $s \mapsto C_\nu(0,s)$ of the Fourier coefficients in Theorem 4 have an explicit description in terms of gamma functions, the zeta function of Riemann and divisor functions; see page 7. The interesting factors are ζ and $\sigma_{2s-1}(|\nu|)$. To single out the interesting part of the C_ν I define in [4, 2.6] functions Ψ_ν by dividing out gamma factors and exponentials. Corollary 2.9 in [4] gives the behavior of Ψ_ν at points $(0,s)$ with $\mathrm{Re}\, s > 1$. (Note that $s_{\text{there}} = s_{\text{here}} - \frac{1}{2}$.)

At points (r,s) with $r \in (0,12)$ and $\mathrm{Re}\, s > 1$ we use the relation between the family E and exponentially growing Poincaré series on page 18 to get the following result:

$$\Psi_\nu(r,s) = \sum_{c=1}^{\infty}\sum_{d \bmod c}{}^{*} e^{2\pi i r S(d,c) + 2\pi i \nu d/c} c^{1-2s} {}_0F_1\left[;2s;-\frac{1}{3}\pi^2 r(\nu + r/12)c^{-2}\right] + O\left(r^{2\,\mathrm{Re}\, s-1}\right) \text{ as } r \downarrow 0, \text{ for } \mathrm{Re}\, s \text{ large},$$

where ${}_0F_1[;c;x] = \sum_{n=1}^{\infty} x^n/(c)_n n!$, and $S(d,c)$ is the Dedekind sum given by

$$S(d,c) = \sum_{x \bmod c}\left(\left(\frac{dx}{c}\right)\right)\left(\left(\frac{x}{c}\right)\right)$$
$$((u)) = \begin{cases} u - [u] - 1/2 & \text{if } 0 < u < 1 \\ 0 & \text{if } u \in \mathbb{Z} \end{cases}$$

See [4, Corollary 2.15, Proposition 2.16].

Taylor expansion. The functions Ψ_ν have a Taylor expansion at $r = 0$ of the form $\Psi_\nu(r,s) = \sum_{n=0}^\infty r^n \Psi_{\nu,n}(s)$. The coefficients $\Psi_{\nu,n}(s)$ are meromorphic in the variable s. It turns out that each $\Psi_{\nu,n}(s)$ with $n \geq 2$ even is holomorphic on a right half plane, with an explicitly known right-most first order singularity at $s = \frac{n+1}{2}$, see [4, Proposition 3.3.].

Proposition 3.5 in [4] gives, for $\mathrm{Re}\, s$ large, an expression for $\Psi_{\nu,n}(s)$ in terms of Dirichlet series of the form

$$\Delta_\nu(k,s) = \sum_{c=1}^{\infty} \sum_{d \bmod c}^{*} c^{-s} \left(\frac{S(d,c)}{c}\right)^k e^{2\pi i \nu d/c}.$$

That expression is complicated, but it can be used to find the rightmost singularity of the $\Delta_\nu(k,s)$. See Proposition 4.2 in [4] (replace $n \leq 2$ in part ii) by $n \geq 2$). Techniques of analytic number theory lead to the following distribution results. (See [4, §5].)

Theorem 6 *Let* $f : [-1,1] \times (\mathbf{R} \bmod \mathbf{Z}) \to \mathbf{C}$ *be continuous. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{c=1}^{N} \frac{1}{c} \sum_{d \bmod c}^{*} f\left(\frac{S(d,c)}{c}, \frac{d}{c}\right) = \frac{6}{\pi^2} \int_0^1 f(0,\tau)\, d\tau.$$

Take $g(\sigma,\tau) = \sigma^2 f(\sigma,\tau)$. *Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{c=1}^{\infty} \sum_{d \bmod c}^{*} g\left(\frac{S(d,c)}{c}, \frac{d}{c}\right) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\varphi((k,n))}{n} \sum_{\pm} g\left(\frac{\pm 1}{12n}, \frac{k}{n}\right).$$

4.5 Annihilation of cusp forms

Theorem 5 gives some information on the question whether cusp forms are annihilated under perturbation of the weight: *Given a cuspidal Maass form of weight 0, does it occur as member of a family of cusp forms with varying weight?*

Perturbation of the group. The modular group Γ_{mod} is rigid in $\mathrm{SL}_2(\mathbf{R})$ (all continuous deformations are given by conjugation). But there are many subgroups that occur as a member of a continuous families of discrete cofinite groups. Consider a family $t \mapsto \Gamma_t$ of such groups. Given a cuspidal Maass form ψ for Γ_0, the question is whether it is the value ψ_0 of a family of cuspidal Maass forms $t \mapsto \psi_t$. This is discussed in [29] and [30]. Phillips and Sarnak say that ψ is *annihilated* under the perturbation if such a family does not exist. They give a necessary condition for annihilation: a certain L-function (depending on ψ) should be non-zero at a certain point (depending on the eigenvalue). Deshouillers and Iwaniec, [8], show that this condition is satisfied for infinitely many cuspidal Maass forms. Wolpert, [38], discusses more complicated deformations of the group, in which the topological type of the quotient changes.

There seems to be the feeling that cuspidal Maass forms are annihilated under "general perturbations".

Perturbation of the multiplier system. For subgroups of the modular groups the space of multiplier systems has in general dimension larger than 1. So in general there are continuous families of multiplier systems suitable for weight 0, i.e., families of group characters. Phillips and Sarnak, [31], discuss families of characters of the subgroup $\Gamma(2)$ of Γ_{mod}, and give a non-vanishing condition for the annihilation of a cuspidal Maass form (for the trivial character) under this perturbation. Again, the results suggest that annihilation is a common phenomenon.

For the modular group there are no non-trivial continuous families of characters. For the family $r \mapsto v_r$ of multiplier systems with varying weight a non-vanishing condition like that in [31] is possible. As far as I know no density result like that in [8] has been proven.

5 Representational point of view

On page 5 I have indicated that modular forms of integral weight can be viewed as functions on $\mathrm{SL}_2(\mathbf{R})$.

Lie algebra action. If one views modular forms of integral weight as functions on $G = \mathrm{SL}_2(\mathbf{R})$ (see page 5) one can let the complexified Lie algebra $\mathbf{g}$ of G act by differentiation on the right. In this way the focus of attention moves from individual modular forms to subspaces of $C^\infty(\Gamma_{\mathrm{mod}}\backslash G)$ in which $\mathbf{g}$ acts. Let f be a modular form with eigenvalue λ, and take $X \in \mathbf{g}$. Then Xf is a linear combination of modular forms with the same eigenvalue λ of the Casimir operator. If we start with a form of weight q, the result is a sum of modular forms of weights $q-2$, q and $q+2$. One has also an action by right translation over elements of the maximal compact subgroup $K := \mathrm{SO}_2(\mathbf{R})$ of G. The action of K preserves the weight.

For non-integral weights similar statements hold for the universal covering group of G, which has the same Lie algebra.

In this way the restriction of the weight to a neighborhood of the interval $(-12, 12)$ in Sections 3 and 4 can be removed. One finds a more general formulation of the results in [3]: Propositions 2.14 and 2.17, and Theorem 2.21.

Automorphic models. For general values of (q, λ) a modular form of weight q and eigenvalue λ generates an irreducible $\mathbf{g}$-module that is isomorphic to the space $H_K^{\xi,\nu}$ of K-finite vectors in a principal series representation. The parameters ξ and ν depend on λ and q mod $2\mathbf{Z}$. So one has a representation of the $\mathbf{g}$-module $H_K^{\xi,\nu}$ in $C^\infty(\Gamma_{\mathrm{mod}}\backslash G)$; this is called an *automorphic model* of $H_K^{\xi,\nu}$. (The name *modular model* would be more appropriate in the context of these lectures, but I do not like its sound in English.) For non-integral weights, we have to work on the universal covering group. To incorporate the multiplier system, we do not work with left-invariant functions, but with functions transforming according to a character.

This is the point of view one finds in [24] and [14]. The operator $\mathbf{M}$ is a map from $H_K^{\xi,\nu}$ to $C^\infty(\Gamma\backslash G)$. The actual Poincaré series we discussed on page 17 are obtained by applying $\mathbf{M}$ to a weight vector in $H_K^{\xi,\nu}$.

In [24] the domain of the operator $\mathbf{M}$ is larger than $H_K^{\xi,\nu}$. It contains the space $H_\omega^{\xi,\nu}$ of analytic vectors in the principal series representation $H^{\xi,\nu}$. The group G itself acts in $H_\omega^{\xi,\nu}$, and the automorphic model $\mathbf{M}$ intertwines this action with right translation in $C^\infty(\Gamma_{\mathrm{mod}}\backslash G)$.

Γ_{mod}-invariant vectors. For the Poincaré series it is not very important to work with $H_\omega^{\xi,\nu}$ instead of $H_K^{\xi,\nu}$. But an automorphic model of $H_\omega^{\xi,\nu}$ can be described by a Γ_{mod}-invariant vector in the contragredient representation $H_{-\omega}^{1/\xi,-\nu}$. In this way the whole $\mathbf{g}$-module of modular forms is described by one Γ_{mod}-invariant object,

which is a hyperfunction on $P\backslash G$, for the standard parabolic subgroup P of G. (A hyperfunction is a linear form on the real analytic functions. This generalizes the notion of a distribution.)

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