J. reine angew. Math. **492** (1997), 1—39

Journal für die reine und angewandte Mathematik

Automorphic forms, hyperfunction cohomology, and period functions

By *Roelof W. Bruggeman* at Utrecht

Abstract. Lewis, [5], has given a relation between cuspidal Maass forms and "period functions". This paper investigates that relation by means of the invariant hyperfunctions corresponding to automorphic forms.

1. Introduction

Lewis, [5], constructs, with help of integral transforms, a bijection between the space of cuspidal Maass forms with eigenvalue $s - s^2$ for the full modular group $\mathrm{PSL}_2(\mathbb{Z})$, and the space of holomorphic functions ψ on $\mathbb{C} \setminus (-\infty, 0]$ that satisfy

$$\psi(z) - \psi(z+1) = (z+1)^{-2s} \psi\left(\frac{z}{z+1}\right), \tag{1}$$

and $\psi(x) = \mathrm{O}(1/x)$ as $x \to \infty$. See also [6].

Lewis, [5], § 6 (c), gives formal computations with the *boundary form* associated to a Maass form as a motivation for his method. The present paper arose from the wish to understand the map from Maass forms to period functions in terms of hyperfunctions.

Almost all real analytic modular forms, cuspidal Maass forms included, arise from invariant hyperfunction vectors in the corresponding principal series representation. These hyperfunctions live on the boundary of the upper half plane, which is topologically a circle. They are represented by holomorphic functions in the upper and lower half plane. Proposition 2.3 shows that a special choice of the representatives always leads to functions that satisfy the relation (1), and that, after a normalization, there is a bijective correspondence between invariant hyperfunctions and solutions of (1), for general values of the spectral parameter.

Zagier, [11], gives arguments why the functions ψ associated to cuspidal Maass forms are similar to the period polynomials associated to holomorphic cusp forms for the modular group. (See also [6], II-C.) We shall show that the classical period polynomials are a specialization of 1-cocycles with values in the hyperfunctions with compact support (see

Proposition 5.8 and the construction in the proof of Proposition 2.5 in Subsection 6.3). To arrive at this cohomological interpretation, we have to work with cocycles on the full original of the modular group in the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, and use hyperfunctions on the line, which arises as the universal covering of the circle bounding the upper half plane.

In Proposition 2.7, we give a natural map from compactly supported hyperfunctions on the line to holomorphic functions on $\mathbb{C}\setminus I$, with $I \subset \mathbb{R}$ an interval. The image of the 1-cocycles mentioned above leads again to Lewis's period functions, but with the opposite choice of the spectral parameter; see Proposition 2.8.

Lewis's period function is a special feature of the modular group. Most results in this paper are valid for a general cofinite discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ with cusps.

Bunke and Olbrich, in [2] and a number of preprints, consider the hyperfunction-valued cohomology for much more general discrete groups. I do not see a direct relation to the cohomology group discussed in this paper.

I thank E.P. van den Ban, J.J. Duistermaat, J.B. Lewis and D. Zagier for their interest, help, and useful discussions.

Many of the ideas in this paper are present in the work of Lewis, or have been the subject of our discussions during Lewis's visits to Utrecht. I have also profited from Lewis's remarks on preliminary versions of this paper. Zagier has brought the work of Lewis to my attention, and has shown interest in the approach in this paper. Van den Ban showed me the argument in the proof of Lemma 5.2. Duistermaat has repeatedly told me that invariant boundary forms should give insight into automorphic forms.

2. Notations and statement of results

2.1. Principal series representations of $\mathbf{PSL_2(\mathbb{R})}$. Let $G := \mathrm{PSL}_2(\mathbb{R})$. Elements of G are indicated by a representative in $\mathrm{SL}_2(\mathbb{R})$. So $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ denote the same element of G.

We use $k(\vartheta) := \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$, and $p(z) := \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}$ for z in the *upper half plane* $\mathfrak{H}^+ = \{x + iy : x \in \mathbb{R}, y > 0\}$. We always write $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$, for $z \in \mathfrak{H}^+$. We also use $n(x) := p(i + x)$ for $x \in \mathbb{R}$ and $a(y) := p(iy)$ for $y > 0$

$N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ is a unipotent subgroup of G, and $A := \{p(iy) : y > 0\}$ a one-dimensional real torus. $P := NA$ is a parabolic subgroup of G. The group

$$K := \{k(\vartheta) : \vartheta \in \mathbb{R} \bmod \pi\mathbb{Z}\}$$

is a maximal compact subgroup of G. As a Haar measure on K we use $dk = \frac{1}{\pi}\, d\vartheta$, with $k = k(\vartheta)$.

Elements of G act on the *complex projective line* $\mathbb{P}^1_{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by *fractional linear transformations* $z \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \dfrac{az+b}{cz+d}$. This action has three orbits: the upper half plane $\mathfrak{H}^+$, the *lower half plane* $\mathfrak{H}^- := \{z \in \mathbb{C} : y < 0\}$, and the *real projective line* $T := \mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

The upper half plane can be viewed as the quotient G/K by identifying gK to $g \cdot i$. We identify T to $P\backslash G$ by $Pk(\vartheta) \mapsto \cot\vartheta$. We provide T with the cyclic order that induces the usual order on $\mathbb{R} \subset T$. Note that the function $\vartheta \mapsto \cot\vartheta$ is *decreasing*. (This choice of the identification avoids some complications in the formulas later on.)

For each $\nu \in \mathbb{C}$, we denote by π_ν the *principal series representation* of G by right translation in the space M^ν of classes of functions $f : G \to \mathbb{C}$ satisfying

$$f(p(z)g) = y^{1/2+\nu} f(g)$$

and $\int_K |f(k)|^2\, dk < \infty$. This gives the induced representation of G corresponding to the character $p(z) \mapsto y^{\nu+1/2}$ of the parabolic subgroup P.

M^ν is the Hilbert space $L^2(K, dk)$ with a G-action π_ν depending on ν. This representation is bounded. The G-spaces (π_ν, M^ν) and $(\pi_{-\nu}, M^{-\nu})$ are dual to each other under the pairing $(f_1, f_2) \mapsto \int_K f_1(k) f_2(k)\, dk$. See, e.g., [4], Chap. III, § 2.

Usually, the letter H is used to indicate these spaces. We employ M to avoid confusion with cohomology groups.

Functions on K correspond to functions on $T \cong P\backslash G$. This gives the realization of M^ν in the functions on T that are square integrable with respect to $\dfrac{d\tau}{\pi(1+\tau^2)}$. The action and the duality are given by:

$$\pi_\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(\tau) = \left(\frac{1+\tau^2}{(c\tau-a)^2 + (d\tau-b)^2} \right)^{1/2+\nu} \varphi\left(\frac{d\tau-b}{-c\tau+a} \right), \tag{2}$$

$$\langle \varphi_1, \varphi_2 \rangle = \frac{1}{\pi} \int_T \varphi_1(\tau)\varphi_2(\tau)\, \frac{d\tau}{1+\tau^2}. \tag{3}$$

The action of G on $T = P\backslash G$ is on the right. If we view T as the real projective line, then this action corresponds to $g : \tau \mapsto g^{-1} \cdot \tau$.

The representation π_ν makes sense in the spaces M^ν_ω and $M^\nu_{-\omega}$, of *analytic vectors* and *hyperfunction vectors* of M^ν. The spaces M^ν_ω and $M^{-\nu}_{-\omega}$ are dual. We have $M^\nu_\omega \subset M^\nu \subset M^\nu_{-\omega}$.

Elements of M^ν_ω are given by holomorphic functions on some neighborhood W of T in $\mathbb{P}^1_{\mathbb{C}}$ (this neighborhood depends on the element). The representation π_ν of G in M^ν_ω is given by (2). Note that $\left(\frac{1+\tau^2}{(c\tau-a)^2+(d\tau-b)^2}\right)^{\nu+1/2}$ is holomorphic in τ on a neighborhood of T depending on $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let W be a neighborhood of T in $\mathbb{P}^1_{\mathbb{C}}$. An element of $M^{\nu}_{-\omega}$ is an equivalence class of holomorphic functions on $W \setminus T$, where two functions are equivalent if their difference extends as a holomorphic function on W. This concept does not depend on the choice of W.

The inclusion $M^{\nu}_{\omega} \subset M^{\nu}_{-\omega}$ associates to a holomorphic function φ on $W \supset T$ the hyperfunction represented by the function equal to φ on $W \cap \mathfrak{H}^+$ and to 0 on $\mathfrak{H}^-$.

The space M^{ν}_{ω} contains the space M^{ν}_{K} of *K-finite vectors*, with algebraic basis $\{\varphi_q : q \in 2\mathbb{Z}\}$, $\varphi_q(k(\vartheta)) = e^{iq\vartheta}$, or in the realization on T: $\varphi_q(\tau) = \left(\frac{\tau+i}{\tau-i}\right)^{q/2}$. This space is not invariant under $\pi_\nu(G)$, but it has the derived action of the complexified Lie algebra $\mathfrak{g}$ of G. It is K-invariant, and $\pi_\nu(k(\vartheta))\,\varphi_q = e^{iq\vartheta}\varphi_q$.

2.2. Automorphic hyperfunctions. By Γ we denote a cofinite discrete subgroup of $G = \mathrm{PSL}_2(\mathbb{R})$. So Γ acts discontinuously on $\mathfrak{H}^+ \cup \mathfrak{H}^-$. The limit set of Γ is equal to T. The quotient $\Gamma \setminus \mathfrak{H}^+$ has finite invariant measure; we assume that it is not compact. The fundamental example in this paper is the modular group $\Gamma_{\mathrm{mod}} := \mathrm{PSL}_2(\mathbb{Z})$.

A *cusp* of Γ is a point of T fixed by an element of Γ of the form $g\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g^{-1}$, with $x \neq 0$, $g \in G$. Let us call $\mathscr{C}(\Gamma)$ the set of cusps of Γ. This non-empty set consists of finitely many Γ-orbits.

We have $\mathscr{C}(\Gamma_{\mathrm{mod}}) = \mathbb{P}^1_{\mathbb{Q}} = \Gamma_{\mathrm{mod}} \cdot \infty$. The group $\Gamma^\infty = \left\{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}$ is the subgroup of Γ_{mod} fixing the cusp ∞. For a general Γ, we arrange by conjugation that $\infty \in \mathscr{C}(\Gamma)$, and that $\{\gamma \in \Gamma : \gamma \cdot \infty = \infty\} = \Gamma^\infty$.

For $\nu \in \mathbb{C}$, let $A^{\nu}_{-\omega}(\Gamma)$ be the space of $\alpha \in M^{\nu}_{-\omega}$ that satisfy $\pi_\nu(\gamma)\alpha = \alpha$ for all $\gamma \in \Gamma$. We call the elements of $A^{\nu}_{-\omega}(\Gamma)$ *automorphic hyperfunctions* for Γ. So $A^{\nu}_{-\omega}(\Gamma) = H^0(\Gamma, M^{\nu}_{-\omega})$.

As an example, consider a *holomorphic automorphic form on Γ of weight* $k \in 2\mathbb{Z}$, i.e., a holomorphic function f on $\mathfrak{H}^+$ such that $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We do not impose any growth condition at the cusps, so k may be negative. We define $\alpha_f \in M^{(k-1)/2}_{-\omega}$ to be the hyperfunction represented by

$$(4) \qquad g_f(\tau) := \begin{cases} (-1)^{k/2} 2^{-k}(1+\tau^2)^{k/2} f(\tau) & \text{for } \tau \in \mathfrak{H}^+, \\ 0 & \text{for } \tau \in \mathfrak{H}^-. \end{cases}$$

The transformation behavior of f under Γ implies that $\alpha_f \in A^{(k-1)/2}_{-\omega}(\Gamma)$.

On the other hand, automorphic hyperfunctions produce automorphic forms by the so called *Poisson integral* (see Thm. 3 of [3]). Let $\nu \in \mathbb{C}$, $q \in 2\mathbb{Z}$. For a given automorphic hyperfunction $\alpha \in A^{\nu}_{-\omega}(\gamma)$, the function

$$F_q(g) := \langle \pi_{-\nu}(g)\varphi_q, \alpha \rangle \tag{5}$$

is an automorphic form on G for Γ of weight q with eigenvalue $\frac{1}{4} - \nu^2$, i.e., it satisfies:

(i) $F_q(\gamma g k(\vartheta)) = F(g) e^{iq\vartheta}$ for all $\gamma \in \Gamma$ and $k(\vartheta) \in K$.

(ii) $\mathbf{C} F_q = \left(\frac{1}{4} - \nu^2\right) F_q$, where $\mathbf{C}$ is the *Casimir operator*.

We do not impose a growth condition at the cusps.

There are differentiation relations between the various F_q. Consider $\mathbf{W} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{E}^{\pm} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pm i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the Lie algebra $\mathfrak{g}$ as left invariant differential operators on G. Then $\mathbf{W} F_q = iq F_q$ and $\mathbf{E}^{\pm} F_q = (1 - 2\nu \pm q) F_{q \pm 2}$.

For $\alpha = \alpha_f$, where f is a holomorphic automorphic form as above, we have $F_k(p(z)) = y^{k/2} f(z)$. (To check this, use (15).) One can show that almost all automorphic forms on G (including those corresponding to cuspidal Maass forms) arise as F_q for some automorphic hyperfunction. In this paper, the automorphic hyperfunctions are the central point of interest, and not the individual automorphic forms.

2.3. Automorphic hyperfunctions and period functions. Let $\mathscr{F}$ be the space of holomorphic functions $f: \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ that satisfy

$$f(\tau) = f(\tau + 1)\,, \tag{6}$$

$$f(\tau) = \mathrm{O}(1) \quad \text{as } |\mathrm{Im}\,\tau| \to \infty\,, \tag{7}$$

$$f(i\infty) + f(-i\infty) = 0\,. \tag{8}$$

Conditions (6) and (7) ensure that f has an expansion in non-negative powers of $e^{\pm 2\pi i \tau}$ on $\mathfrak{H}^{\pm}$. So $f(\pm i\infty)$ makes sense as the constant term in this expansion.

Let $T_0 = T \setminus \{\infty\}$. So $T_0 = \mathbb{R}$ in the coordinate τ. Any hyperfunction α on T has a restriction $\alpha|_{T_0}$ to T_0, which can be represented by a holomorphic function on $\{\tau \in \mathbb{C} : 0 < |\mathrm{Im}\,\tau| < 1\}$.

Proposition 2.1. *There is an injective linear map $\alpha \mapsto f_\alpha : A^{\nu}_{-\omega}(\Gamma) \to \mathscr{F}$, such that $\tau \mapsto (1+\tau^2)^{1/2+\nu} f_\alpha(\tau)$ represents the restriction $\alpha|_{T_0}$.*

We define powers of $1 + \tau^2$ with the standard choice $\arg(1+\tau^2) \in (-\pi, \pi]$. So $\tau \mapsto (1+\tau^2)^{1/2+\nu}$ is discontinuous on $i[1,\infty) \cup (-i)[1,\infty)$.

For the formulation of this result, it is essential that ∞ is a cusp of Γ, with width 1.

Example. Let $f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\tau}$ be a holomorphic automorphic form of weight $k \in 2\mathbb{Z}$, and let α_f be the corresponding automorphic hyperfunction, represented by g_f as in (4). Then

$$f_\alpha(\tau) = 2^{-k}(-1)^{k/2} \cdot \begin{cases} \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau} & \text{for } \tau \in \mathfrak{H}^+, \\ -\frac{1}{2} a_0 - \sum_{n=1}^{\infty} a_{-n} e^{-2\pi i n\tau} & \text{for } \tau \in \mathfrak{H}^-. \end{cases}$$

Example. Let $\nu \notin \frac{1}{2} + \mathbb{Z}$ and $\alpha \in A^{\nu}_{-\omega}(\gamma)$. Suppose that the automorphic form F_0 of weight zero corresponding to α (see (5)) does not vanish, and has the Fourier expansion

$$F_0(p(z)) = \sum_{n \neq 0} c_n e^{2\pi i n x} W_{0,\nu}(4\pi|n|y).$$

($W_{\kappa,s}$ is the exponentially decreasing Whittaker function.) Proposition 4.6 shows that the function f_α in Proposition 2.1 satisfies

$$f_\alpha(\tau) = \begin{cases} \pi^{\nu+1/2}\Gamma(1/2-\nu) \sum_{n=1}^{\infty} n^{\nu+1/2} c_n e^{2\pi i n\tau} & \text{for } \tau \in \mathfrak{H}^+, \\ -\pi^{\nu+1/2}\Gamma(1/2-\nu) \sum_{n=1}^{\infty} n^{\nu+1/2} c_{-n} e^{-2\pi i n\tau} & \text{for } \tau \in \mathfrak{H}^-. \end{cases}$$

Equations (11) and (19) in [6] show that our f_α is $\frac{1}{4}\pi^{\nu+1/2}\Gamma(1/2-\nu)$ times the function f in Theorem 3 of [6].

The map $\alpha \mapsto f_\alpha$ is far from surjective. In the modular case we can describe the image:

Proposition 2.2. *Let $\nu \in \mathbb{C}$. Define $\mathscr{F}_{\mathrm{mod}}(\nu)$ to be the subspace of $f \in \mathscr{F}$ for which $\tau \mapsto f(\tau) - \tau^{-1-2\nu} f(-1/\tau)$ extends holomorphically to $\mathbb{C} \setminus (-\infty, 0]$.*

The image of $A^{\nu}_{-\omega}(\Gamma_{\mathrm{mod}}) \to \mathscr{F} : \alpha \mapsto f_\alpha$ is equal to $\mathscr{F}_{\mathrm{mod}}(\nu)$.

Let $\nu \in \mathbb{C}$. We define $\Psi_{\mathrm{mod}}(\nu)$ to be the linear space of holomorphic functions $\psi : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ that satisfy

$$\psi(\tau) = \psi(\tau+1) + (\tau+1)^{-2\nu-1}\psi\left(\frac{\tau}{\tau+1}\right), \tag{9}$$

$$e^{-\pi i\nu} \lim_{\operatorname{Im}\tau \to \infty} \left(\psi(\tau) + \tau^{-1-2\nu}\psi\left(\frac{-1}{\tau}\right)\right) + e^{\pi i\nu} \lim_{\operatorname{Im}\tau \to -\infty} \left(\psi(\tau) + \tau^{-1-2\nu}\psi\left(\frac{-1}{\tau}\right)\right) = 0. \tag{10}$$

The existence of both limits is part of condition (10). Equation (9) is Lewis's period relation (1), with $2s$ replaced by $2\nu+1$.

Proposition 2.3. *Let $v \in \mathbb{C}$. For $\alpha \in A^{v}_{-\omega}(\Gamma_{\mathrm{mod}})$, put*

$$(11) \qquad \psi_\alpha(\tau) := f_\alpha(\tau) - \tau^{-1-2v} f_\alpha(-1/\tau)\,,$$

with f_α as in Proposition 2.1. *The linear map $\alpha \mapsto \psi_\alpha : A^{v}_{-\omega}(\Gamma_{\mathrm{mod}}) \to \Psi_{\mathrm{mod}}(v)$ is bijective if $v \notin \frac{1}{2} + \mathbb{Z}$.*

This shows that all hyperfunctions associated to automorphic forms for the modular group (without any growth condition at the cusp) lead to solutions of Lewis's functional equation (1).

If α is an automorphic hyperfunction associated to a holomorphic automorphic form for Γ_{mod}, then $\psi_\alpha = 0$; so the injectivity fails for $v \in \frac{1}{2} + \mathbb{Z}$.

In the case that $\alpha \in A^{v}_{-\omega}(\Gamma_{\mathrm{mod}})$ corresponds to a Maass form, our ψ_α is

$$\frac{1}{4}\,\pi^{v+1/2}\,\Gamma(1/2 - v)$$

times Lewis's period function ψ.

The relation (11) and its inverse relation (17) are due to Lewis see [6], II-B. We shall prove the Propositions 2.1, 2.2 and 2.3 in Section 3.2.

2.4. Γ-decompositions. Let $X \subset T$ be a non-empty Γ-invariant set. We define a *Γ-decomposition q* on X of an automorphic hyperfunction $\alpha \in A^{v}_{-\omega}(\Gamma)$ as a map $(\xi, \eta) \mapsto \alpha[\xi, \eta]_q$ that assigns to pairs $(\xi, \eta) \in X^2$ with $\xi \neq \eta$ an element of $M^{v}_{-\omega}$ according to the following rules:

(a) $\mathrm{Supp}(\alpha[\xi, \eta]_q) \subset [\xi, \eta]$ for all $\xi, \eta \in X$, $\xi \neq \eta$.

The interval $[\xi, \eta]$ refers to the natural cyclic order on T. The hyperfunction $\alpha[\xi, \eta]_q$ is required to have restriction 0 on (η, ξ).

(b) $\sum_{j=1}^{n} \alpha[\xi_{j-1}, \xi_j]_q = \alpha$ whenever $\xi_0, \xi_1, \ldots, \xi_n \in X$ satisfy $\xi_0 < \xi_1 < \cdots < \xi_n = \xi_0$, and the intervals $[\xi_{j-1}, \xi_j]$ intersect only in their end points.

(c) $\alpha[\gamma \cdot \xi, \gamma \cdot \eta]_q = \pi_v(\gamma)\alpha[\xi, \eta]_q$ for all $\gamma \in \Gamma$, $\xi, \eta \in X$, $\xi \neq \eta$.

The nature of hyperfunctions admits breaking up all hyperfunctions α such that (a) and (b) are satisfied. The Γ-behavior in condition (c) is crucial in this definition.

We define $A^{v}_{-\omega}(\Gamma, X)$ to be the linear space of $\alpha \in A^{v}_{-\omega}(\Gamma)$ that have a Γ-decomposition on X. The next result, to be proved in Subsection 5.2, gives conditions under which Γ-decompositions exist.

Theorem 2.4. *Let $X \subset T$ be a Γ-invariant non-empty subset of T, without hyperbolic fixed points of Γ.*

If $\nu \in \mathbb{C}$, $\nu \neq \frac{1}{2} + \mathbb{Z}_{\geqq 0}$, then $A^{\nu}_{-\omega}(\Gamma, X) = A^{\nu}_{-\omega}(\Gamma)$.

If $\nu \in \frac{1}{2} + \mathbb{Z}_{\geqq 0}$, then $A^{\nu}_{-\omega}(\Gamma, X)$ consists of those $\alpha \in A^{\nu}_{-\omega}(\Gamma)$ that satisfy $A^{\xi}_0(\alpha) = 0$ for all $\xi \in X \cap \mathscr{C}(\Gamma)$.

Here $A^{\infty}_0(\alpha)$ is the constant term in the expansion of f_α on $\mathfrak{H}^+$ as a function of $e^{2\pi i \tau}$, see Proposition 2.1. If $\xi \in \mathscr{C}(\Gamma)$, then $A^{\xi}_0(\alpha) := A^{\infty}_0\big(\pi_\nu(g)\alpha\big)$ for any $g \in G$ with $g \cdot \infty = \xi$ such that $g \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g^{-1}$ generates the subgroup of Γ fixing ξ. The condition $A^{\xi}_0(\alpha) = 0$ needs to be checked only for one ξ in each Γ-orbit in $X \cap \mathscr{C}(\Gamma)$.

For automorphic hyperfunctions α such that the automorphic forms F_q in (5) are cusp forms, we shall define, in Proposition 5.7, the *geodesic decomposition* gd of α on $\mathscr{C}(\Gamma)$. For cusps ξ and η, we shall give $\alpha[\xi, \eta]_{\mathrm{gd}}$ by an integral from ξ to η in $\mathfrak{H}^+$, and one from η to ξ in $\mathfrak{H}^-$, over paths that approach ξ and η along geodesics for the hyperbolic metric.

This is similar to the *period polynomials* associated to holomorphic cusp forms f of weight $k \in 2\mathbb{Z}$:

$$(12) \qquad R_f(\xi, \eta; X) := \int_{\xi}^{\eta} f(\tau)(X - \tau)^{k-2}\, d\tau \quad \text{for } \xi, \eta \in \mathscr{C}(\Gamma)\,.$$

In Subsection 4.1, we shall see that there is a morphism of G-modules from $M^{(k-1)/2}_{-\omega}$ to the polynomials of degree at most $k-2$. If $\alpha_f \in A^{(k-1)/2}_{-\omega}(\Gamma)$ is the hyperfunction corresponding to f, then this morphism sends $\alpha[\xi, \eta]_{\mathrm{gd}}$ to $\frac{1}{4\pi} R_f(\xi, \eta)$, see Proposition 5.8.

2.5. Universal covering group. The period polynomial $R_f(\cdot, \cdot; X)$ is a homogeneous 1-cocycle of the type discussed in Subsection 6.2, with values in the polynomials of degree at most $k-2$. But, from $\alpha[\xi, \eta]_q + \alpha[\eta, \xi]_q = \alpha$, we see that a Γ-decomposition is not a 1-cocycle (unless $\alpha = 0$).

This is the reason why we go over to the universal covering group $\tilde{G}$ of G. (See Subsection 6.1.) The action of G on T lifts to an action of $\tilde{G}$ on the line $\tilde{T}$. The principal series representations π_ν of G correspond to representations $\tilde{\pi}_\nu$ of $\tilde{G}$ on functions and hyperfunctions on the line $\tilde{T}$. In particular, we have the representation $\tilde{\pi}_\nu$ of $\tilde{G}$ acting in the space of hyperfunctions $\tilde{M}^{\nu}_{-\omega}$. It leaves invariant the subspace $\tilde{M}^{\nu}_{-\omega,c}$ of hyperfunctions with compact support. By $\tilde{\Gamma}$ we denote the full original of Γ in $\tilde{G}$. Automorphic hyperfunctions $\alpha \in A^{\nu}_{-\omega}(\Gamma)$ correspond to $\tilde{\Gamma}$-invariant elements of $\tilde{M}^{\nu}_{-\omega}$.

Proposition 2.5. *Let X be a non-empty, Γ-invariant subset of X. Denote by $\tilde{X}$ its full original in $\tilde{T}$. Let $\nu \in \mathbb{C}$.*

For $\alpha \in A^{\nu}_{-\omega}(\Gamma, X)$ and each Γ-decomposition q of α, there is a homogeneous 1-cocycle $c_{\alpha,q}$ on $\tilde{X}$ with values in $\tilde{M}^{\nu}_{-\omega,c}$ such that for all $\xi, \eta \in \tilde{X}$, $\xi < \eta$, the restriction of $c_{\alpha,q}(\xi, \eta)$ to (ξ, η) is equal to the restriction to (ξ, η) of the element of $(\tilde{M}^{\nu}_{-\omega})^{\tilde{\Gamma}}$ corresponding to α, and $\mathrm{Supp}(c_{\alpha,q}(\xi, \eta)) \subset [\xi, \eta]$.

The class of $c_{\alpha,q}$ in $H^1_{\tilde{X}}(\tilde{\Gamma}, \tilde{M}^{\nu}_{-\omega,c})$ does not depend on q. The resulting map

$$A^{\nu}_{-\omega}(\Gamma, X) \to H^1_{\tilde{X}}(\tilde{\Gamma}, \tilde{M}^{\nu}_{-\omega,c})$$

is injective. Its image consists of those classes that can be represented by a homogeneous 1-cocycle satisfying $\mathrm{Supp}(c(\xi, \eta)) \subset [\xi, \eta]$ *for all $\xi, \eta \in \tilde{X}$, $\xi < \eta$.*

Subsection 6.2 gives a discussion of the cohomology groups $H^1_{\tilde{X}}(\tilde{\Gamma}, A) \subset H^1(\tilde{\Gamma}, A)$, for right $\tilde{\Gamma}$-modules A. In the case that $X = \mathscr{C}(\Gamma)$, the cohomology group $H^1_{\tilde{X}}(\tilde{\Gamma}, A)$ consists of those classes that can be represented by an inhomogeneous 1-cocycle $\eta : \tilde{\Gamma} \to A$ with the property that for each $\xi \in \tilde{X}$ there exists $a_\xi \in A$ such that $\eta(\gamma) = a_\xi \gamma - a_\xi$ for all $\xi \in \tilde{\Gamma}$ that leave fixed ξ. We call this group $H^1_{\tilde{X}}(\tilde{\Gamma}, A)$ the *first parabolic cohomology group* on $\tilde{\Gamma}$ with values in A.

Theorem 2.6. *Let $\nu \in \mathbb{C}$. There is an injective linear map $A^{\nu}_{-\omega}(\Gamma) \to H^1(\tilde{\Gamma}, \tilde{M}^{\nu}_{-\omega,c})$. If $\nu \notin \frac{1}{2} + \mathbb{Z}_{\geqq 0}$, then the image is contained in the first parabolic cohomology group on $\tilde{\Gamma}$ with values in $\tilde{M}^{\nu}_{-\omega,c}$.*

We prove these results in Subsection 6.3.

2.6. Action in holomorphic functions of weight $1-2\nu$. There is a standard lifting $G \to \tilde{G} : g \mapsto \tilde{g}$. This is used to describe the action of $\tilde{G}$ in the functions on $\mathfrak{H}^+ \cup \mathfrak{H}^-$ of complex weight r:

$$F|_r \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z) = (cz+d)^{-r} F\left(\frac{az+b}{cz+d}\right) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

with $-\pi < \arg(ci + d) < \pi$. If the weight r is an even integer, then the center of $\tilde{G}$ acts trivially, and we have the usual action of G of weight r.

By $\mathscr{H}_r$ we denote the space of holomorphic functions on $\mathfrak{H}^+ \cup \mathfrak{H}^-$, provided with this action. Some elements of $\mathscr{H}_r$ may have a holomorphic extension to an interval contained in $\mathbb{R}$. Lewis's functional equation (1) can be written as $\psi = \psi|_{2s} \widetilde{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} + \psi|_{2s} \widetilde{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$.

Proposition 2.7. *There is an intertwining operator $\mathsf{P} : \tilde{M}^{\nu}_{-\omega,c} \to \mathscr{H}_{1-2\nu}$. If $\alpha \in \tilde{M}^{\nu}_{-\omega,c}$ has support contained in $[\xi, \eta]$ with $-\pi < \eta < \xi \leqq 0$, then $\mathsf{P}\alpha$ has a holomorphic extension to the interval* $(\cot \eta, \infty)$.

Here we use the coordinate ϑ on $\tilde{T}$ that corresponds to the one-parameter subgroup $\vartheta \mapsto \tilde{k}(\vartheta)$ in $\tilde{G}$ covering $\vartheta \mapsto k(\vartheta)$ in G.

The operator P induces maps $H^i_{\tilde X}(\tilde\Gamma, \tilde M^n_{-\omega,c}) \to H^i_{\tilde X}(\tilde\Gamma, \mathscr{H}_{1-2\nu})$ for each non-empty $\tilde\Gamma$-invariant $\tilde X \subset \tilde T$. In the next result, we assume that $\tilde X$ contains the full original of $\mathscr{C}(\Gamma)$.

Proposition 2.8. *Let $\nu \notin \frac{1}{2} + \mathbb{Z}$, and let $\alpha \in A^\nu_{-\omega}(\Gamma)$ be such that all corresponding automorphic forms F_q (see (5)) are cusp forms. Then*

$$\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi, 0) = \frac{-i}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}-\nu\right)\Gamma(1-\nu)^{-1}\left(f_{\iota(\nu)\alpha} - f_{\iota(\nu)\alpha}|_{1-2\nu}\tilde k(-\pi)\right), \tag{13}$$

where $\iota(\nu) : M^\nu_{-\omega} \to M^{-\nu}_{-\omega}$ is the isomorphism given in (18), *and $f_{\iota(\nu)\alpha}$ is the function given in Proposition* 2.1 *for the automorphic hyperfunction $\iota(\nu)\alpha \in A^{-\nu}_{-\omega}(\Gamma)$.*

If moreover, $\Gamma = \Gamma_{\mathrm{mod}}$, then

$$\mathsf{P}c_{\alpha,\mathrm{gd}}\left(-\frac{\pi}{2}, 0\right) = \frac{-i}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}-\nu\right)\Gamma(1-\nu)^{-1}\psi_{\iota(\nu)\alpha}\,. \tag{14}$$

This gives a cohomological interpretation of Lewis's period function.

We prove these results in Section 7.

3. Hyperfunctions

3.1. Representatives of hyperfunctions. The manifold $T \cong P\backslash G$ has real dimension 1. That makes hyperfunctions on T reasonably concrete objects. We discuss hyperfunctions, first on $\mathbb{R}$, and next on T. For more information, and for proofs of the following facts, see [8], § 1.1–3.

Holomorphic functions. Let $\mathscr{O}$ be the sheaf of holomorphic functions on the complex projective line $\mathbb{P}^1_{\mathbb{C}}$. So $\mathscr{O}(U)$ is the space of holomorphic functions on U for each open $U \subset \mathbb{P}^1_{\mathbb{C}}$.

The sheaf $\mathscr{A}$ of real analytic functions on $\mathbb{R}$ is the restriction $\mathscr{O}|_{\mathbb{R}}$. This expresses the fact that each real analytic function on an open set $U \subset \mathbb{R}$ is the restriction of a holomorphic function on some open set $V \subset \mathbb{P}^1_{\mathbb{C}}$ containing U.

Hyperfunctions on $\mathbb{R}$. Let $U \subset \mathbb{R}$ be open in $\mathbb{R}$, and take $V \subset \mathbb{P}^1_{\mathbb{C}}$ open in $\mathbb{P}^1_{\mathbb{C}}$, $V \supset U$. The linear space $\mathscr{B}(U) := \mathscr{O}(V \setminus U) \bmod \mathscr{O}(V)$ does not depend on the choice of V. Elements of $\mathscr{B}(U)$ are called *hyperfunctions* on U.

Intuitively, a hyperfunction on U, represented by $g \in \mathscr{O}(V \setminus U)$, is the jump in g when we cross U.

The presheaf $U \mapsto \mathscr{B}(U)$ is a sheaf on $\mathbb{R}$.

Support of an hyperfunction. The *support* $\mathrm{Supp}(\mu)$ of $\mu \in \mathscr{B}(U)$ is the smallest closed set $S \subset U$ such that the restriction of μ in $\mathscr{B}(U \setminus S)$ is equal to 0.

Duality. Let $U \subset \mathbb{R}$ be open, and $S \subset U$ be compact. There is a duality between $\mathscr{A}(U)$ and the space of $\alpha \in \mathscr{B}(U)$ with $\mathrm{Supp}(\alpha) \subset S$ given by $\langle \varphi, \alpha \rangle := \int_C \varphi(\vartheta) g(\vartheta) \frac{d\vartheta}{\pi}$, where C is a contour encircling S once in positive direction, inside the intersection of the domain of φ and the domain of the representative g of α. (This intersection is not necessarily connected. Use the component that has S in its closure.)

Hyperfunctions on T. Locally, T looks like $\mathbb{R}$. We define the sheaf $\mathscr{B}_T$ of hyperfunctions on T in the same way as $\mathscr{B}$. So $\mathscr{B}_T(U) = \mathcal{O}(V \setminus U) \bmod \mathcal{O}(V)$, independent of the choice of an open neighborhood V of U inside $\mathbb{P}^1_{\mathbb{C}}$.

Duality. There is a duality between the real analytic functions on $U \subset T$ and the hyperfunctions on U with compact support. We shall mainly use the case $U = T$:

$$\langle f, \alpha \rangle := \int_{C_+} f(\tau) g(\tau) \frac{d\tau}{\pi(1+\tau^2)} + \int_{C_-} f(\tau) g(\tau) \frac{d\tau}{\pi(1+\tau^2)}, \tag{15}$$

for $f \in \mathcal{O}(W)$, $g \in \mathcal{O}(W \setminus T)$ a representative of $\alpha \in \mathscr{B}_T(T)$, and $C_\pm$ contours contained in $W \setminus T$; see Figure 1. If α has support inside an interval $I \neq T$ inside T, we can rewrite the sum of both integrals as a new integral over a curve encircling I in *negative* direction. Note that $\frac{d\vartheta}{\pi}$ corresponds to $-\frac{d\tau}{\pi(1+\tau^2)}$ under $\tau = \cot \vartheta$.

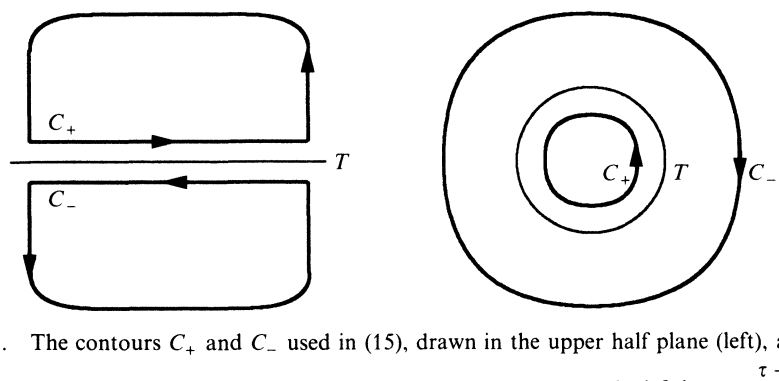


Figure 1. The contours C_+ and C_- used in (15), drawn in the upper half plane (left), and in the circle model (right). The coordinate w on the right is related to the coordinate τ on the left by $w = \frac{\tau - i}{\tau + i} = e^{-2i\vartheta}$. The contours at the left and the right are not equal, but homotopic.

The set $W \supset T$ is not drawn. The contours should be adapted to the set W in such a way that the region between the contours and T is contained in W.

Real analytic functions and hyperfunctions on T. The sheaf $\mathscr{A}_T := \mathcal{O}|_T$ is the sheaf of real analytic functions on T. We define an injection $\mathscr{A}_T(T) \mapsto \mathscr{B}_T(T)$ by sending $f \in \mathcal{O}(U)$ for an open $U \subset \mathbb{P}^1_{\mathbb{C}}$, $U \supset T$, to the hyperfunction represented by $g(\tau) = f(\tau)$ for $\tau \in \mathfrak{H}^+ \cap U$ and $g(\tau) = 0$ for $\tau \in \mathfrak{H}^-$. With this interpretation, (15) is an extension of the duality in (3).

For $\varphi_q \in \mathscr{A}_T(T)$, $\varphi_q(\tau) = \left(\frac{\tau + i}{\tau - i}\right)^{q/2}$, $q \in 2\mathbb{Z}$, we prefer to represent $\varphi_q \in \mathscr{B}_T(T)$ in the following way:

	$q < 0$	$q = 0$	$q > 0$
on $\mathfrak{H}^+$	$\left(\frac{\tau + i}{\tau - i}\right)^{q/2}$	$\frac{1}{2}$	0
on $\mathfrak{H}^-$	0	$-\frac{1}{2}$	$-\left(\frac{\tau + i}{\tau - i}\right)^{q/2}$

Representative. The hyperfunction $\alpha \in \mathscr{B}_T(T)$ can be recovered from the linear form $\varphi \mapsto \langle \varphi, \alpha \rangle$ on the real analytic functions. For each $\tau_0 \in \mathbb{C} \setminus \mathbb{R}$ the function

$$h_{\tau_0} : \tau \mapsto \frac{-i}{2} \frac{1 + \tau_0 \tau}{\tau - \tau_0}$$

is holomorphic on $\mathbb{P}^1_{\mathbb{C}} \setminus \{\tau_0\}$. One can check that $g(\tau_0) := \langle h_{\tau_0}, \alpha \rangle$ is the unique representative of α that is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies $g(i) + g(-i) = 0$.

This unique representative can be written in the form $\sum_{q \in 2\mathbb{Z}} c_q \varphi_q$ with $c_q = \mathrm{O}(p^{|q|})$ as $|q| \to \infty$, for each $p > 1$. If we take the representatives of $\varphi_q \in \mathscr{B}_T(T)$ as indicated above, then the corresponding series actually converges on $\mathfrak{H}^+ \cup \mathfrak{H}^-$. The duality amounts to $\langle \sum_q d_q \varphi_q, \sum_q c_q \varphi_q \rangle = \sum_q c_{-q} d_q$.

Hyperfunction vectors. Let $M^{\nu}_{-\omega}$ be the space $\mathscr{B}_T(T)$ of hyperfunctions on T, provided with the action π_ν of G described by (2). In this way we have an inclusion of $\pi_\nu(G)$-spaces $M^{\nu}_{\omega} \subset M^{\nu} \subset M^{\nu}_{-\omega}$.

The duality satisfies $\langle \pi_{-\nu}(g)\varphi, \pi_\nu(g)\alpha \rangle = \langle \varphi, \alpha \rangle$. So $M^{\nu}_{-\omega}$ is dual to $M^{-\nu}_{\omega}$.

3.2. Proofs of Propositions 2.1, 2.2 and 2.3.

Proof of Proposition 2.1. Take a representative g of α. The function $F : \tau \mapsto (1 + \tau^2)^{-1/2-\nu} g(\tau)$ is holomorphic on the strip $0 < |\mathrm{Im}\,\tau| < \varepsilon$ for some $\varepsilon \in (0, 1)$. The invariance of α under $\pi_\nu \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ implies that $F(\tau - 1) = F(\tau) + q(\tau)$, with q holomorphic on $|\mathrm{Im}\,\tau| < \varepsilon$. So F represents a hyperfunction on $\mathbb{R}$ that is invariant under the translations $\tau \mapsto \tau + 1$. It determines a hyperfunction on the circle $\mathbb{R} \bmod \mathbb{Z}$, and this hyperfunction has a representative that is holomorphic on the complement of the circle in $\mathbb{P}^1_{\mathbb{C}}$. The freedom in this representative is an additive constant. We replace F by the unique function f_α of the form

$$f_\alpha(\tau) = \begin{cases} \frac{1}{2} A_0(\alpha) + \sum_{n=1}^{\infty} A_n(\alpha) e^{2\pi i n \tau} & \text{for } \tau \in \mathfrak{H}^+, \\ -\frac{1}{2} A_0(\alpha) - \sum_{n=1}^{\infty} A_{-n}(\alpha) e^{-2\pi i n \tau} & \text{for } \tau \in \mathfrak{H}^-. \end{cases} \tag{16}$$

So $f_\alpha \in \mathscr{F}$, and $\tau \mapsto (1 + \tau^2)^{1/2+\nu} f_\alpha(\tau)$ represents $\alpha|_{T_0}$.

Suppose that f_α vanishes. Then $\mathrm{Supp}(\alpha) \subset \{\infty\}$. As ∞ cannot be a fixed point of the whole group Γ, we conclude that $\alpha = 0$. $\square$

Remark. Equation (16) defines the coefficients $A_n(\alpha)$ for each automorphic hyperfunction α. They may be viewed as a kind of Fourier coefficients.

For any cusp $\xi \in \mathscr{C}(\Gamma)$, we have defined $A_0^\xi(\alpha) = A_0\big(\pi_\nu(g_\xi)^{-1}\alpha\big)$, with $g_\xi \in G$ such that $g_\xi \cdot \infty = \xi$ and $g_\xi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g_\xi^{-1}$ generates the subgroup of Γ fixing ξ. Note that if $n \neq 0$, the analogous definition of $A_n^\xi(\alpha)$ would depend on the choice of g_ξ.

Proof of Proposition 2.2. Let $\alpha \in A^\nu_{-\omega}(\Gamma_{\mathrm{mod}})$. The restriction to $T \backslash \{0\}$ of

$$\pi_\nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \alpha = \alpha$$

has representative $\tau \mapsto (1+\tau^{-2})^{1/2+\nu} f_\alpha(-1/\tau)$. The gluing condition on $(0,\infty)$ between this representative and $\tau \mapsto (1+\tau^2)^{1/2+\nu} f_\alpha(\tau)$ is the condition characterizing $\mathscr{F}_{\mathrm{mod}}(\nu)$ inside $\mathscr{F}$. The gluing conditions on $(-\infty, 0)$ and on $(0,\infty)$ are equivalent.

Conversely, let $f \in \mathscr{F}_{\mathrm{mod}}(\nu)$. The function $\tau \mapsto (1+\tau^2)^{1/2+\nu} f(\tau)$ represents a hyperfunction $\beta_0 \in \mathscr{B}_T(T_0)$ that satisfies $\pi_\nu \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \beta_0 = \beta_0$. Put $\beta_\infty := \pi_\nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \beta_0 \in \mathscr{B}_T(T \backslash \{0\})$. This hyperfunction has representative $\tau \mapsto (1+\tau^{-2})^{1/2+\nu} f(-1/\tau)$. As $f \in \mathscr{F}_{\mathrm{mod}}(\nu)$, the gluing conditions on $(0,\infty)$ and $(-\infty,0)$ are satisfied. So β_0 and β_∞ are restrictions of a $\pi_\nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$-invariant $\beta \in \mathscr{B}_T(T)$. The support of $\pi_\nu \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \beta - \beta$ is contained in $\{\infty\}$. On $|\tau| > 2$, $\tau \notin \mathbb{R}$, this hyperfunction is represented by

$$\begin{aligned} &\left(\frac{\tau^2+1}{1+(\tau-1)^2}\right)^{1/2+\nu} (1+(\tau-1)^{-2})^{1/2+\nu} f\left(\frac{-1}{\tau-1}\right) - (1+\tau^{-2})^{1/2+\nu} f\left(\frac{-1}{\tau}\right) \\ &\quad = (1+\tau^{-2})^{1/2+\nu} \left(\left(\frac{\tau}{\tau-1}\right)^{1+2\nu} f\left(\frac{-1}{\tau-1} - 1\right) - f\left(\frac{-1}{\tau} + 1\right) \right) \\ &\quad = -(1+\tau^{-2})^{1/2+\nu} \left(f\left(\frac{\tau-1}{\tau}\right) - \left(\frac{\tau-1}{\tau}\right)^{-1-2\nu} f\left(-1/\frac{\tau-1}{\tau}\right) \right). \end{aligned}$$

As $f \in \mathscr{F}_{\mathrm{mod}}(\nu)$, the quantity between brackets is holomorphic on a neighborhood of $\dfrac{\tau-1}{\tau} = 1$. This shows that $\pi_\nu \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \beta - \beta$ vanishes on a neighborhood of ∞.

As $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate Γ_{mod}, this implies $\beta \in A^\nu_{-\omega}(\Gamma_{\mathrm{mod}})$. $\square$

Proof of Proposition 2.3. Proposition 2.2 implies that ψ_α satisfies (9). If $\nu \in \frac{1}{2} + \mathbb{Z}$, then $\psi_\alpha(\tau) + \tau^{-1-2\nu}\psi_\alpha(-1/\tau) = 0$, and (10) is satisfied. If $\nu \notin \frac{1}{2} + \mathbb{Z}$, we recover Lewis's inversion formula:

$$(17) \qquad f_\alpha(\tau) = \frac{1}{1 + e^{\mp 2\pi i \nu}} \left(\psi_\alpha(\tau) + \tau^{-1-2\nu}\psi_\alpha(-1/\tau)\right) \quad \text{for } \tau \in \mathfrak{H}^\pm\,,$$

which implies (10). □

4. Behavior of automorphic hyperfunctions at a cusp

The proof of Theorem 2.4 will be based on knowledge of the behavior of automorphic hyperfunctions at a cusp. This can be compared to the study of automorphic forms with help of their Fourier expansion. We do not give the complete Fourier series. The expansion at the cusp discussed in Subsection 4.3 suffices for our purpose. The crucial term in this expansion corresponds to the Fourier term of order zero. This term is an N-invariant hyperfunction. In Subsection 4.2, we construct explicitly a basis of the corresponding space $(M^\nu_{-\omega})^N$. To see that we really have a basis, we use the results in Subsection 4.1 on the structure of M^ν_K.

4.1. The $(\mathfrak{g}, K)$-modules M^ν_K. The complexified Lie algebra $\mathfrak{g}$ of G acts in M^ν_K. For $\mathbf{W} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{E}^\pm = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pm i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$d\pi_\nu(\mathbf{W})\,\varphi_q = iq\,\varphi_q \quad \text{and} \quad d\pi_\nu(\mathbf{E}^\pm)\,\varphi_q = (1 + 2\nu \pm q)\,\varphi_{q\pm 2}\,.$$

Irreducible case. The $(\mathfrak{g}, K)$-spaces M^ν_K are irreducible if $\nu \notin \frac{1}{2} + \mathbb{Z}$, see, e.g., [4], Chap. VI, § 5. For these ν, there are isomorphisms $\iota(\nu) : M^\nu_K \to M^{-\nu}_K$ defined by

$$(18) \qquad \iota(\nu)\,\varphi_q := \left(\frac{1}{2} - \nu\right)_{|q/2|} \left(\frac{1}{2} + \nu\right)^{-1}_{|q/2|} \varphi_q\,,$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. The extensions

$$\iota(\nu) : M^\nu_\omega \to M^{-\nu}_\omega \quad \text{and} \quad \iota(\nu) : M^\nu_{-\omega} \to M^{-\nu}_{-\omega}$$

intertwine the G-actions π_ν and $\pi_{-\nu}$.

Reducible case. Let $\nu = \dfrac{k-1}{2}$, $k \in 2\mathbb{Z}$, $k \geqq 2$.

$M^{(k-1)/2}_{-\omega}$ has invariant subspaces D^k_+ and D^k_-, spanned by the φ_q with $q \geqq k$, respectively $q \leqq -k$. The quotient $M^{(k-1)/2}_{-\omega} \bmod D^k_+ \oplus D^k_-$ has finite dimension, and is isomorphic to an irreducible subspace of $M^{-(k-1)/2}_K$.

This quotient can also be viewed as the space of polynomials of degree at most $k-2$, with the right action of G given by $p|_{2-k}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(X) = (cX+d)^{k-2} p\left(\frac{aX+b}{cX+d}\right)$. To construct an explicit intertwining operator, put $h_X^{\langle k \rangle}(\tau) := -(2i)^{k-2}(\tau^2+1)^{1-k/2}(\tau - X)^{k-2}$. This is a polynomial in X with coefficients in $M_\omega^{(1-k)/2}$. Define for $\alpha \in M_{-\omega}^{(k-1)/2}$:

$$\alpha^{\langle k \rangle} := \langle h_X^{\langle k \rangle}, \alpha \rangle . \tag{19}$$

A computation shows that:

$$\left(\pi_\nu(g)\alpha\right)^{\langle k \rangle} = \alpha^{\langle k \rangle}|_{2-k}\, g^{-1} \quad \text{for all } g \in G .$$

Intertwining operators. Let $\nu \in \mathbb{C}$ and $\alpha \in M_{-\omega}^{\nu}$. We define $T_\alpha : M_K^{-\nu} \to C^\infty(G)$ by $T_\alpha \varphi(g) := \langle \pi_{-\nu}(g)\varphi, \alpha \rangle$. This operator intertwines the $(\mathfrak{g}, K)$-action in $M_K^{-\nu}$ with the action in $C^\infty(G)$ by right differentiation and translation. If α is invariant under $\pi_\nu(X)$ for some subgroup $X \subset G$, then the functions in $T_\alpha(M_K^{-\nu})$ are left-invariant under X.

4.2. N-invariant hyperfunctions. We determine a basis of the space

$$X(\nu) := \{\alpha \in M_{-\omega}^{\nu} : \pi_\nu(n(x))\alpha = \alpha \text{ for all } x \in \mathbb{R}\} = (M_{-\omega}^{\nu})^N$$

for each $\nu \in \mathbb{C}$ by constructing sufficiently many linearly independent elements in it.

Intertwining operators in the left-N-invariant functions. We recall the following description of the space of intertwining operators $M_K^{-\nu} \to C^\infty(N \backslash G)$. The dimension of this space is an upper bound for $\dim X(\nu)$.

(a) For all $\nu \in \mathbb{C}$, there is the standard form of $M_K^{-\nu}$ as induced representation from P to G. It is given by $\mathrm{St}(-\nu)\varphi_q(p(z)k(\vartheta)) := y^{1/2-\nu} e^{iq\vartheta}$.

(b) For $\nu \notin \frac{1}{2} + \mathbb{Z}$, $\nu \neq 0$, a basis of the space of representations in $C^\infty(N \backslash G)$ is given by $\mathrm{St}(-\nu)$ and $\mathrm{St}(\nu)\iota(-\nu)$. See (18) for the isomorphism $\iota(-\nu)$.

(c) Also in the case $\nu = 0$, the space of such representations has dimension 2. There is a representation L, not a multiple of $\mathrm{St}(0)$, with $L\varphi_q(p(z)k(\vartheta)) = y^{1/2}(l_q + \log y)e^{iq\vartheta}$. (This does not determine L completely.)

(d) If $\nu \in \frac{1}{2} + \mathbb{Z}$, $\nu < 0$, then the space of representations has dimension 2. A basis is given by $\mathrm{St}(-\nu)$ and $S(-\nu)$, where $S(-\nu)\varphi(g) := \lim_{\mu \to \nu} \mathrm{St}(\mu)\iota(-\mu)\varphi(g)$.

(e) If $\nu \in \frac{1}{2} + \mathbb{Z}$, $\nu > 0$, then there is a basis $\mathrm{St}(-\nu)$, $S_+(-\nu)$, $S_-(-\nu)$, with $S_+(-\nu)\varphi(g) := \lim_{\mu \to \nu} (\mu - \nu)\mathrm{St}(\mu)\iota(-\mu)\varphi(g)$, $S_-(-\nu)\varphi_q := \mathrm{sign}(q) S_+(\nu)\varphi_q$.

N-invariant hyperfunctions. In the sequel, we exhibit elements of $X(v)$ and indicate the corresponding representation in $C^\infty(N\backslash G)$. Then we consult the list (a)–(e) to see that we have a basis of $X(v)$. We leave to the reader most of the computations needed to check the statements. The invariance of α under $\pi_v(N)$ is a consequence of the left-N-invariance of the $T_\alpha \varphi_q$.

Within $X(v)$ we define the subspace $X_\infty(v) := \{\alpha \in X(v) : \operatorname{Supp}(\alpha) \subset \{\infty\}\}$.

A hyperfunction for $\operatorname{St}(-v)$. The function $\tau \mapsto \frac{-i}{2}\tau$ determines $\delta \in X_\infty(v)$ for all $v \in \mathbb{C}$, such that $T_\delta = \operatorname{St}(-v)$.

A hyperfunction for a multiple of $\operatorname{St}(v)\,\iota(-v)$. Let $v \in \mathbb{C}$, $v \notin \mathbb{Z}_{<0}$, and put

$$p(\tau) := \begin{cases} \sqrt{\pi}\,\Gamma(-v)^{-1}(1+\tau^2)^{v+1/2} & \text{on } \mathfrak{H}^+ \backslash i[1,\infty), \\ 0 & \text{on } \mathfrak{H}^-, \end{cases}$$

$$q(\tau) := \tau^{1+2v}(1+\tau^{-2})^{v+1/2}\,\Gamma(1+v)\,e^{-\pi i v \operatorname{sign}(\operatorname{Im}\tau)}/(2i\sqrt{\pi}) \quad \text{for } \tau \notin \mathbb{R} \cup i[-1,1]\,.$$

We define $\mu(v)$ as the hyperfunction represented by p on T_0, and by q on $T\backslash\{0\}$. (The gluing relations are satisfied.) For $\varphi \in M_\omega^{-v}$ we have:

$$\langle \varphi, \mu(v)\rangle = \int_{A_+} \frac{\varphi(\tau)p(\tau)\,d\tau}{\pi(1+\tau^2)} + \sum_{\pm}\int_{B_\pm} \frac{\varphi(\tau)q(\tau)\,d\tau}{\pi(1+\tau^2)}, \tag{20}$$

with contours as indicated in Figure 2. This is holomorphic in $v \in \mathbb{C}\backslash\mathbb{Z}_{<0}$. For $\operatorname{Re} v < 0$ we move off the contours $B_\pm$ to infinity, and obtain

$$\langle \varphi, \mu(v)\rangle = \int_{-\infty}^{\infty} \varphi(\tau)p(\tau)\,\frac{d\tau}{\pi(1+\tau^2)}. \tag{21}$$

We use this integral to define $\mu(v)$ for $v \in \mathbb{Z}_{<0}$. A computation with $\varphi = \pi_{-v}(a(y))\varphi_q$ shows that $T_{\mu(v)} = \Gamma\left(\frac{1}{2}-v\right)^{-1}\operatorname{St}(v)\,\iota(-v)$ for $v \notin \frac{1}{2}+\mathbb{Z}$ (check this first for $\operatorname{Re} v < 0$, then extend the relation by holomorphy). As $v \mapsto \langle\varphi, \mu(v)\rangle$ is holomorphic on $\mathbb{C}$, we have

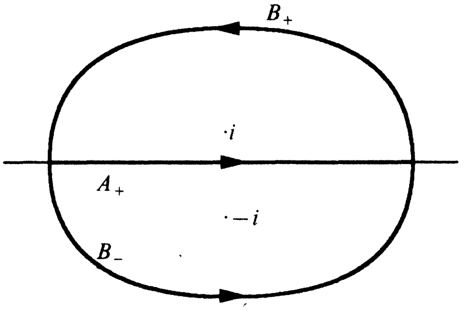


Figure 2. Contours for the computation of $\langle \varphi, \mu(v)\rangle$ in (20).

$T_{\mu(\nu)} = \Gamma\left(\frac{1}{2} - \nu\right)^{-1} S(-\nu)$ for $\nu \in \frac{1}{2} + \mathbb{Z}$, $\nu < 0$, and $T_{\mu(\nu)} = (-1)^{b/2}\left(\frac{b}{2} - 1\right)! S_+(-\nu)$ for $\nu = \frac{b-1}{2}$, $b \in 2\mathbb{Z}_{>0}$.

The restriction of $\mu(\nu)$ to T_0 is represented by $\sqrt{\pi}\,\Gamma(-\nu)^{-1}(1+\tau^2)^{1/2+\nu}$ on $\mathfrak{H}^+ \setminus i[1,\infty)$ and $\tau \mapsto 0$ on $\mathfrak{H}^-$. For $\nu \notin \mathbb{Z}_{<0}$ this is clear. For $\nu \in \mathbb{Z}_{<0}$, we use (21) with $\varphi(\tau) = \frac{-i}{2}\frac{1+\tau\tau_0}{\tau - \tau_0}$.

Support $\{\infty\}$. If $\nu \in \mathbb{Z}_{\geqq 0}$, then $\mu(\nu) \in X_\infty(\nu)$.

Let $\nu \in \mathbb{Z}_{>0}$. A representative of $\mu(\nu)$ on a neighborhood of ∞ is

$$q(\tau) = (-1)^\nu (2i\sqrt{\pi})^{-1}\,\nu!\,\tau^{1+2\nu}(1+\tau^{-2})^{\nu+1/2},$$

which is clearly not a representative of a multiple of δ. As $\dim X(\nu) \leqq 2$, we conclude that $X(\nu) = X_\infty(\nu)$ for $\nu \in \mathbb{Z}_{>0}$.

Case $\nu = 0$. We have $\mu(0) = \pi^{-1/2}\delta$. Put $\lambda(\nu) := \frac{1}{\nu}\left(\mu(\nu) - \pi^{-1/2}\delta\right)$ for $0 < |\nu| < \frac{1}{2}$. For each φ holomorphic on a neighborhood of T and each $g \in G$, the function $\nu \mapsto \langle \pi_{-\nu}(g)\varphi, \lambda(\nu)\rangle$ is holomorphic at $\nu = 0$. A computation shows that

$$\lim_{\nu \to 0} \langle \pi_{-\nu}\big(p(z)\big)\varphi_q, \lambda(\nu)\rangle = \frac{2}{\sqrt{\pi}}\, y^{1/2}(\log y + \ell_q),$$

with $\ell_q = \mathrm{O}(\log|q|)$ as $|q| \to \infty$. Hence, this determines a hyperfunction $\lambda(0) \in (M^0_{-\omega})^N$. The N-invariance extends to $\nu = 0$, and $T_{\lambda(0)}$ is linearly independent of $\mathrm{St}(0)$. For the restriction of $\lambda(\nu)$ to T_0 we find a representative that is equal to $-\sqrt{\pi}(1+\tau^2)^{\nu+1/2}$ on $\{\tau \in \mathbb{C} : 0 < \operatorname{Im}\tau < 1\}$ and vanishes on $\mathfrak{H}^-$.

Case $\nu \in \frac{1}{2} + \mathbb{Z}$, $\nu > 0$. A third element of $X_\infty(\nu)$, linearly independent of δ and $\mu(\nu)$, is represented by $\tau \mapsto (1+\tau^2)^{\nu+1/2}$ on a neighborhood of ∞.

Lemma 4.1. $\dim X(\nu) = 3$ *if* $\nu \in \frac{1}{2} + \mathbb{Z}_{>0}$, *and equals* 2 *for other* $\nu \in \mathbb{C}$.

If $\nu \in \mathbb{Z}_{>0}$, *then* $X_\infty(\nu) = X(\nu)$, *otherwise* $X_\infty(\nu)$ *has codimension* 1 *in* $X(\nu)$.

For each $\nu \in \mathbb{C} \setminus \mathbb{Z}_{>0}$, *there exists* $\kappa_\nu \in X(\nu)$ *for which the restriction to* T_0 *is represented by* $\tau \mapsto (1+\tau^2)^{\nu+1/2}$ *on* $\mathfrak{H}^+ \setminus i[1,\infty)$, *and* $\tau \mapsto 0$ *on* $\mathfrak{H}^-$.

Proof. We have shown most statements. We choose

$$\kappa_\nu := \begin{cases} \pi^{-1/2}\Gamma(-\nu)\mu(\nu) & \text{if } \nu \in \mathbb{C}\setminus\mathbb{Z}_{\geqq 0}, \\ -\pi^{-1/2}\lambda(0) & \text{if } \nu = 0. \end{cases} \tag{22}$$

We can add any element of $X_\infty(\nu)$ to κ_ν. The present choice works well in the sequel. □

4.3. Expansion at ∞. Proposition 2.1 attaches to each invariant hyperfunction α a holomorphic function f_α on $\mathbb{C}\setminus\mathbb{R}$; equation (16) gives the expansion of f_α in terms of coefficients $A_n(\alpha)$. In this subsection, we use f_α to give an expansion of α based on the behavior at the cusp ∞.

Lemma 4.2. *For each $\alpha \in A^\nu_{-\omega}(\Gamma)$, there exists $\alpha^c \in (M^\nu_{-\omega})^{\Gamma_\infty}$ for which the restriction to T_0 is represented by the function $\tau \mapsto f^c_\alpha(\tau)(1+\tau^2)^{\nu+1/2}$ on*

$$(\mathfrak{H}^+ \setminus i[1,\infty)) \cup (\mathfrak{H}^- \setminus (-i)[1,\infty)),$$

with

$$f^c_\alpha(\tau) := \begin{cases} \sum_{n=1}^{\infty} A_n(\alpha) e^{2\pi i n\tau} & \text{for } \tau \in \mathfrak{H}^+, \\ -\sum_{n=1}^{\infty} A_{-n}(\alpha) e^{-2\pi i n\tau} & \text{for } \tau \in \mathfrak{H}^-. \end{cases} \tag{23}$$

Remark. The conditions do not determine α^c uniquely.

Proof. Let us take contours $I_\pm$ as indicated in Figure 3, and define α^c as the hyperfunction represented by

$$g^c_\alpha(\tau_0) := \frac{1}{2\pi i}\sum_{\pm}\int_{I_\pm} \frac{1+\tau\tau_0}{\tau-\tau_0} f^c_\alpha(\tau)(1+\tau^2)^{\nu-1/2}\,d\tau. \tag{24}$$

We have to adapt the contours to the position of τ_0, such that τ_0 is inside I_+ or inside I_-. Note that f^c_α has exponential decay.

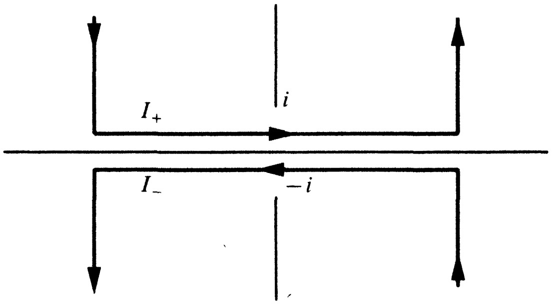


Figure 3. The contours I_+ and I_-.

For $\tau_0 \in \mathfrak{H}^\varepsilon$, $\varepsilon \in \{+, -\}$, τ_0 between I_ε and the real axis, we have

$$\begin{aligned} g_\alpha^c(\tau_0) &= (1+\tau_0^2)^{\nu+1/2} f_\alpha^c(\tau_0) \\ &\quad + \frac{1}{2\pi i} \sum_\pm \int_{I_\pm} \frac{1}{\tau - \tau_0} f_\alpha^c(\tau)(1+\tau^2)^{\nu+1/2}\, d\tau \\ &\quad - \frac{1}{2\pi i} \sum_\pm \int_{I_\pm} f_\alpha^c(\tau)\tau(1+\tau^2)^{\nu-1/2}\, d\tau\,. \end{aligned}$$

The terms with integrals are holomorphic in τ_0 on a neighborhood of T_0. So $\tau \mapsto f_\alpha^c(\tau)(1+\tau^2)^{\nu+1/2}$ represents $\alpha^c|_{T_0}$. A computation shows that for $\varphi \in M_\omega^{-\nu}$

$$\langle \varphi, \alpha^c \rangle = \frac{1}{\pi} \sum_\pm \int_{I_\pm} \varphi(\tau) f_\alpha^c(\tau)(1+\tau^2)^{\nu-1/2}\, d\tau\,, \tag{25}$$

with $I_\pm$ inside the domain of φ. The 1-periodicity of f_α^c gives $\pi_\nu\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\alpha^c = \alpha^c$. □

Fourier term. Let g be a representative of $\alpha \in A^\nu_{-\omega}(\Gamma)$. We define, for $m \in \mathbb{Z}$,

$$\mathsf{F}_m g(\tau) := \int_{x=0}^{1} e^{2\pi i m x} \pi_\nu \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g(\tau)\, dx\,.$$

This defines $\mathsf{F}_m g$ on a neighborhood of T that is a bit smaller than the domain of g. The hyperfunction represented by $\mathsf{F}_m g$ does not depend on the choice of g. We call this hyperfunction $\mathsf{F}_m \alpha$. We call it the Fourier term of α of order m. It transforms under $\pi_\nu(N)$ according to the character $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto e^{-2\pi i m x}$.

We also define $\mathsf{F}_m \beta$ for Γ^∞-invariant hyperfunctions on T_0. Restriction to T_0 commutes with F_m.

For $q \in 2\mathbb{Z}$, and $\alpha \in A^\nu_{-\omega}(\Gamma)$, the m-th term in the Fourier expansion of the automorphic form F_q in (5) is $g \mapsto \langle \pi_{-\nu}(g)\varphi_q, \mathsf{F}_m \alpha \rangle$.

The hyperfunction $\mathsf{F}_0 \alpha$ is invariant under $\pi_\nu(N)$. This is the only Fourier term that we study explicitly.

Proposition 4.3. *For each $\alpha \in A^\nu_{-\omega}(\Gamma)$ there are $\alpha^0, \alpha^\infty \in (M^\nu_{-\omega})^{\Gamma^\infty}$, such that*

(i) $\alpha = \alpha^0 + \alpha^\infty$,

(ii) *the restriction $\alpha^0|_{T_0}$ has $\tau \mapsto (1+\tau^2)^{\nu+1/2} f_\alpha(\tau)$ as a representative,*

(iii) $\mathrm{Supp}(\alpha^\infty) \subset \{\infty\}$.

If $\nu \in \mathbb{Z}_{>0}$, then $A_0(\alpha) = 0$ for each $\alpha \in A^\nu_{-\omega}(\Gamma)$.

Remark. This does not determine α^0 and α^∞ uniquely.

Proof. Consider first $\mathsf{F}_0\alpha$, for $\alpha \in A^{\nu}_{-\omega}(\Gamma)$, $\nu \in \mathbb{Z}_{>0}$. It is an element of $X(\nu) = X_\infty(\nu)$, see Lemma 4.1, so $\mathsf{F}_0\alpha|_{T_0} = 0$. Compute $\mathsf{F}_0(\alpha|_{T_0})$ with help of Proposition 2.1 to see that $A_0(\alpha) = 0$.

Take $\alpha^0 := \alpha^c$ if $\nu \in \mathbb{Z}_{>0}$, and $\alpha^0 := \alpha^c + A_0(\alpha)\kappa_\nu$ otherwise. The restriction of α^0 has the right form, as we see from Proposition 2.1, Lemma 4.1, and Lemma 4.2. This also gives the $\pi_\nu(\Gamma^\infty)$-invariance. The restriction of $\alpha^\infty := \alpha - \alpha^0$ to T_0 vanishes. □

4.4. Cuspidal automorphic hyperfunctions.

Definition 4.4. We call a real analytic automorphic form F for Γ *cuspidal at* ∞, if it is given by a Fourier expansion of the form

$$F(p(z)k(\vartheta)) = \sum_{n \neq 0} a_n(f) e^{2\pi i n x} W_{q\,\mathrm{sign}(n)/2,\nu}(4\pi|n|y) e^{iq\vartheta}, \tag{26}$$

where $q \in 2\mathbb{Z}$ is the weight and $\frac{1}{4} - \nu^2 \in \mathbb{C}$ the eigenvalue, and where $W_{\cdot,\cdot}$ is the exponentially decreasing Whittaker function, see, e.g., [10], 1.7.

We define F to be *cuspidal* at $\xi \in \mathscr{C}(\Gamma)$ if $g \mapsto F(g_\xi g)$ is cuspidal at ∞. (As before, $g_\xi \in G$ is chosen such that $\xi = g_\xi \cdot \infty$, and such that $g_\xi n(1) g_\xi^{-1}$ generates the subgroup of Γ fixing ξ.)

An automorphic form F is a *cusp form*, if it is cuspidal at all $\xi \in \mathscr{C}(\Gamma)$.

Spectral parameter. The action of the Lie algebra preserves cuspidality at a point ξ. Cusp forms are square integrable on $\Gamma\backslash G$. This puts conditions on the $(\mathfrak{g}, K)$-module generated by a cusp form, see [4], Chap. VI, § 6. Such a module is isomorphic to either M_K^ν with $\nu \in i\mathbb{R} \cup \left(-\frac{1}{2}, \frac{1}{2}\right)$, or to an irreducible quotient of $M_K^{(1-k)/2}$, with $k \in 2\mathbb{Z}_{\geqq 1}$. The trivial representation, which is a quotient of $M_K^{1/2}$, occurs in $L^2(\Gamma\backslash G)$, but as the space of constant functions; hence it does not consist of cusp forms.

If Γ has more than one cuspidal orbit, an automorphic form can be cuspidal at some cusps, and not at others. For such an automorphic form, the spectral parameter does not necessarily satisfy the conditions mentioned above.

Definition 4.5. We call an automorphic hyperfunction $\alpha \in A^{\nu}_{-\omega}(\Gamma)$ *cuspidal at* ∞ if $\alpha = \alpha^c$. We call α cuspidal at $\xi \in \mathscr{C}(\Gamma)$ if $\pi_\nu(g_\xi)^{-1}\alpha$ is cuspidal at ∞.

We call α *cuspidal*, if it is cuspidal at all cusps $\xi \in \mathscr{C}(\Gamma)$.

Proposition 4.6. *An automorphic hyperfunction $\alpha \in A^{\nu}_{-\omega}(\Gamma)$ is cuspidal at $\xi \in \mathscr{C}(\Gamma)$, if and only if all associated automorphic forms $F_q(g) = \langle \pi_{-\nu}(g)\varphi_q, \alpha \rangle$ are cuspidal at ξ.*

If α is cuspidal at ∞, then $a_n(F_q)$, as defined in (26), *is related to the coefficients $A_n(\alpha)$, defined in* (16), *by*

$$a_n(F_q) = \frac{(-1)^{q/2}(\pi|n|)^{-\nu-1/2}}{\Gamma\left(\frac{1}{2}-\nu+\frac{1}{2}q\,\mathrm{sign}(n)\right)} A_n(\alpha)\,. \tag{27}$$

Proof. Compute $\langle \pi_{-\nu}(p(z))\varphi_q, \alpha^c\rangle$ by means of (25), and interchange the order of summation and integration. This gives

$$\langle \pi_{-\nu}(g)\varphi_q, \alpha^c\rangle = \sum_{n\neq 0} A_n(\alpha) p_n(g,\nu)\,,$$

$$p_n(g,\nu) := \frac{\mathrm{sign}(n)}{\pi} \int_{I_{\mathrm{sign}(n)}} \pi_{-\nu}(g)\varphi_q(\tau) e^{2\pi i n\tau}(1+\tau^2)^{\nu-1/2}\,d\tau\,.$$

The quantity $p_n(g,\nu)$ is holomorphic in ν. We compute it for $\mathrm{Re}\,\nu<0$, by deforming the contour $I_{\mathrm{sign}(n)}$ into the real axis:

$$\begin{aligned} p_n(p(z),\nu) &= \frac{1}{\pi} y^{1/2-\nu} \int_{-\infty}^{\infty} \left(\frac{1+\tau^2}{y^2+(\tau-x)^2}\right)^{1/2-\nu} \left(\frac{\tau-\bar z}{\tau-z}\right)^{q/2} e^{2\pi i n\tau}(1+\tau^2)^{\nu-1/2}\,d\tau \\ &= \frac{(-1)^{q/2}(\pi|n|)^{-\nu-1/2}}{\Gamma\left(\frac{1}{2}-\nu+\frac{1}{2}q\,\mathrm{sign}(n)\right)} e^{2\pi i n x} W_{q\,\mathrm{sign}(n)/2,\nu}(4\pi|n|y)\,. \end{aligned}$$

So if α is cuspidal at ∞, then we obtain the full Fourier expansion of each F_q. If, on the other hand, all F_q are cuspidal at ∞, then $\langle \varphi_q, \alpha-\alpha^c\rangle = 0$ for all $q\in 2\mathbb{Z}$. This shows that $\alpha=\alpha^c$. $\square$

Corollary 4.7. *Let $\nu\notin \frac{1}{2}+\mathbb{Z}$. If $\alpha\in A^{\nu}_{-\omega}(\Gamma)$ is cuspidal at ∞, then*

$$A_n(\iota(\nu)\alpha) = (\pi|n|)^{-2\nu}\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)^{-1} A_n(\alpha) \quad \textit{for } n\in\mathbb{Z}, n\neq 0\,.$$

Proof. If $\nu\notin\frac{1}{2}+\mathbb{Z}$, the element φ_0 generates the $(\mathfrak{g},K)$-module $M_K^{-\nu}$. So α is determined by the cusp form of weight zero $F_0(g) = \langle \pi_{-\nu}(g)\varphi_0, \alpha\rangle$. Similarly, $\iota(\nu)\alpha$ is determined by $\tilde F_0(g) = \langle \pi_\nu(g)\varphi_0, \iota(\nu)\alpha\rangle$. This is equal to

$$\langle \iota(\nu)\pi_\nu(g)\varphi_0, \alpha\rangle = \langle \pi_{-\nu}(g)\iota(\nu)\varphi_0, \alpha\rangle = \langle \pi_{-\nu}(g)\varphi_0,\alpha\rangle = F_0(g)\,.$$

From (26), we conclude that $a_n(\tilde F_0) = a_n(F_0)$ (use $W_{\kappa,\nu} = W_{\kappa,-\nu}$). $\square$

5. Decompositions

5.1. Partings. To prove that Γ-decompositions exist, we first consider how to decompose hyperfunctions locally.

Fix group. If X is a subgroup of G and $\xi\in T$, we define $X^\xi := \{g\in X : g\cdot\xi = \xi\}$.

So $N^\infty = N$, and $N^\xi = \{1\}$ if $\xi \in T_0 = T \setminus \{\infty\}$. The subgroup $\Gamma^\xi \subset \Gamma$ is trivial for most $\xi \in T$. The exceptions are the *cusps* $\xi \in \mathscr{C}(\Gamma)$ and the *hyperbolic fixed points*. For the latter, Γ^ξ is generated by an element of the form $g_\xi a(t) g_\xi^{-1}$, with $g_\xi \in G$ and $t > 1$. For a cusp ξ, the group Γ^ξ is generated by $g_\xi n(1) g_\xi^{-1}$.

Definition 5.1. Let $a \in \mathbb{R}$ and let $\alpha \in \mathscr{B}(U)$ be a hyperfunction on a neighborhood U of a. We call a *parting* of α at a a decomposition $\alpha = \alpha_+ + \alpha_-$, where $\alpha_\pm \in \mathscr{B}(U)$ are such that $\mathrm{Supp}(\alpha_+) \subset [a, \infty) \cap U$ and $\mathrm{Supp}(\alpha_-) \subset (-\infty, a] \cap U$.

Any parting of α on a smaller neighborhood $U_1 \subset U$ of a, gives rise to a parting on U. So a parting of α at a is a decomposition of the germ of α in the stalk $\mathscr{B}_a$.

A parting is far from unique. We can replace $\alpha_\pm$ by $\alpha_\pm + \beta$ for any hyperfunction β with $\mathrm{Supp}\,\beta = \{a\}$.

Lemma 5.2. *Any $\alpha \in \mathscr{B}(U)$, U open, has a parting at each $a \in U$.*

Proof. The sheaf of hyperfunctions $\mathscr{B}$ is flasque, i.e., the restriction maps $\mathscr{B}(V) \to \mathscr{B}(U)$ are surjective for all open $U \subset V \subset \mathbb{R}$ (see [8], page 2).

Let $a \in (p, q) \subset U$. The sheaf properties imply that there is a hyperfunction β on $(p, a) \cup (a, q)$ such that β has restriction 0 on (p, a), and coincides with α on (a, p). The flasqueness implies that there exists $\alpha_+ \in \mathscr{B}(p, q)$ with restriction β on $(p, a) \cup (a, q)$. So $\mathrm{Supp}(\alpha_+) \subset [a, p)$, and $\alpha_- = \alpha - \alpha_+$ has support contained in $(q, a]$. □

Partings on T. As T is locally isomorphic to $\mathbb{R}$, the order structure included, the concept of parting can be transferred to germs of hyperfunctions on T.

For a parting of a hyperfunction α at $\tau_0 \in T$, we define α_+ and α_- with respect to the cyclic ordering of T. If $\alpha \in (\mathscr{B}_T)_{\tau_0}$ corresponds to $\tilde{\alpha} \in \mathscr{B}_{\vartheta_0}$ under the decreasing map $\vartheta \mapsto \cot \vartheta$, then $\tilde{\alpha}_\pm$ corresponds to $\alpha_\mp$.

Definition 5.3. Let X be a subgroup of G. By an *X-parting* of α at $\xi \in T$ we mean a parting $\alpha = \alpha_+ + \alpha_-$ such that $\pi_\nu(g)\alpha_\pm = \alpha_\pm$ for all $g \in X^\xi$.

Note that the fix group X^ξ acts in the stalk $(\mathscr{B}_T)_\xi$. Any parting at ξ is an X-parting if X^ξ is trivial. If a hyperfunction α has an X-parting at ξ, then $\pi_\nu(g)\alpha = \alpha$ in the stalk $(\mathscr{B}_T)_\xi$ for each $g \in X^\xi$.

Proposition 5.4. *Let $\alpha \in A^\nu_{-\omega}(\Gamma)$. Consider a cusp $\xi \in \mathscr{C}(\Gamma)$.*

If $\nu \notin \frac{1}{2} + \mathbb{Z}$, then α has a Γ-parting at ξ.

If $\nu \in \frac{1}{2} + \mathbb{Z}$, then α has a Γ-parting at ξ if and only if $A_0^\xi(\alpha) = 0$.

Proof. It suffices to consider this at the cusp ∞.

We use the expansion $\alpha = \alpha^0 + \alpha^\infty$ in Proposition 4.3. As α^∞ has support inside $\{\infty\}$, and is invariant under the fix group Γ^∞, it has the Γ-parting $\alpha^\infty = \alpha^\infty + 0$. In Lemma 5.6 we shall see that α^c has a Γ-parting at ∞. So we are left with $A_0(\alpha)\kappa_\nu$. This is treated in Lemma 5.5. □

Lemma 5.5. *Let $\nu \in \mathbb{C} \setminus \mathbb{Z}_{>0}$. Then κ_ν, as defined in* (22), *has a Γ-parting at ∞ if and only if $\nu \notin \frac{1}{2} + \mathbb{Z}_{\geqq 0}$.*

Proof. We shall show that κ_ν has an N-parting at ∞ if $\nu \notin \frac{1}{2}\mathbb{Z}_{>0}$, and has no Γ-parting at ∞ if $\nu \in \frac{1}{2} + \mathbb{Z}_{>0}$. We leave to the reader many computations needed to check the statements in this proof.

Let $\nu \in \mathbb{C}$, $\nu \notin \frac{1}{2} + \mathbb{Z}$, $\nu \notin \mathbb{Z}_{\leqq 0}$. The functions q_+ and q_-, given by

$$q_\pm(\tau) = \frac{\pm\Gamma(1+\nu)}{4i\sqrt{\pi}\cos\pi\nu}(\pm\tau)^{1+2\nu}(1+\tau^{-2})^{\nu+1/2}$$

with the arguments of $\pm\tau$ and $1+\tau^{-2}$ in $(-\pi, \pi)$, represent hyperfunctions

$$\mu_\pm(\nu) \in \mathscr{B}_T(1, -1)$$

(cyclic order) that determine a parting of $\mu(\nu)$ at ∞. They turn out to be invariant under N in the stalk $(\mathscr{B}_T)_\infty$.

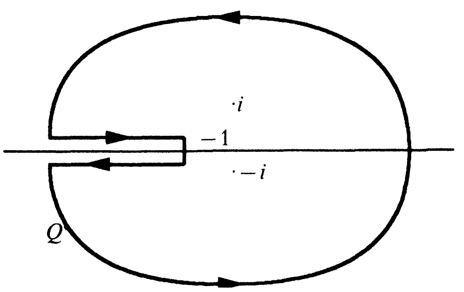


Figure 4. Contour used in the definition of $\tilde{\mu}_+(\nu)$ in (28).

Consider the contour Q in Figure 4, and define $\tilde{\mu}_+(\nu) \in M^{\nu}_{-\omega}$ for $\nu \notin \frac{1}{2} + \mathbb{Z}$, $\nu \notin \mathbb{Z}_{\leqq 0}$, by

$$(28) \qquad \langle \varphi, \tilde{\mu}_+(\nu)\rangle := \int_Q \varphi(\tau) q_+(\tau) \frac{d\tau}{\pi(1+\tau^2)}.$$

By taking $\varphi(\tau) = \frac{-i}{2}\frac{1+\tau\tau_0}{\tau-\tau_0}$, with τ_0 inside Q, one checks that $\tilde{\mu}_+(\nu)$ has image $\mu_+(\nu)$ in the stalk $(\mathscr{B}_T)_\infty$. If $\operatorname{Re}\nu < 0$, then we can move off the contour to ∞, and find

$$\langle \varphi, \tilde{\mu}_+(\nu)\rangle = \int_{-\infty}^{-1} \varphi(\tau) p(\tau + 0i) \frac{d\tau}{\pi(1+\tau^2)}.$$

We use this integral to define $\mu_+(v)$ for all v with $\operatorname{Re} v < 0$, and $\mu_-(v) := \mu(v) - \mu_+(v)$. This turns out to define an N-parting at ∞.

Take $\lambda_\pm(v) := \frac{1}{v}\left(\mu_\pm(v) - \frac{1}{2}\pi^{-1/2}\delta\right)$. This determines an N-parting of $\lambda(v)$ at ∞ for $v \neq 0$, which extends to $v = 0$.

Suppose that $v = \frac{b-1}{2}$, with $b \in 2\mathbb{Z}_{>0}$. Up to a non-zero multiple, the hyperfunction $\mu(v)$ has representative $\tau \mapsto (1+\tau^2)^{b/2}\operatorname{sign}(\operatorname{Im}\tau)$ in $\mathcal{O}(\mathbb{C}\setminus\mathbb{R})$. For any parting of $\mu(v)$ at ∞, a representative r of $\mu_+(v)$ has to be holomorphic on a set $R < |\tau| < \infty$, $\tau \notin (-\infty, -R)$, and there has to be a holomorphic extension across $(-\infty, -R)$ satisfying

$$r(x+0i) - r(x-0i) = 2(1+x^2)^{b/2}$$

for $x < -R$. Hence $r(\tau) = (1+\tau^2)^{b/2}\left(\frac{\log\tau}{\pi i} + r_1(\tau)\right)$, with r_1 holomorphic on $R < |\tau| < \infty$.

Invariance under $\pi_v\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ implies that we have, for large $|\tau|$:

$$\frac{1}{\pi i}\log(1 - x/\tau) + r_1(\tau - x) - r_1(\tau) \equiv 0 \bmod \tau^{-b}.$$

If we expand $\log(1 - x/\tau)$ and $r_1(\tau)$ in Laurent series in τ, we see that the terms τ^n with $n \geqq 0$ do not interact with the terms with $n < 0$. So we leave out the sum with non-negative powers of τ, and assume that r_1 is holomorphic at $\tau = \infty$, with a zero at ∞. We fix $x > 0$. For $\tau < -R$, we have $r_1(\tau - x) - r_1(\tau) = -\frac{1}{\pi i}\log(1 - x/\tau) + \mathrm{O}_x(\tau^{-b}) = \frac{x}{\pi i}\frac{1}{\tau} + \mathrm{O}_x(\tau^{-2})$. For each $k \in \mathbb{Z}_{>0}$, we find $r_1(\tau - kx) = \frac{x}{\pi i}\sum_{m=0}^{k-1}\frac{1}{\tau - mx} + \mathrm{O}_x(1/\tau)$. This contradicts $r_1(\tau - kx) = \mathrm{O}\big(1/(\tau - kx)\big) = \mathrm{O}_x(1/k)$. □

Lemma 5.6. *Let $\alpha \in A^v_{-\omega}(\Gamma)$. The hyperfunction α^c defined in Lemma 4.2 has a Γ-parting at ∞.*

Proof. In (24), we have given a representative of α^c, by integration over the contours $I_\pm$ in Figure 3. We take half of the contours $I_\pm$, with end point $\frac{1}{2}i$, respectively, initial point $-\frac{1}{2}i$; see Figure 5. We define a function $g_\alpha^{c,\ell}$, holomorphic on $|\tau_0| > 1$, $\tau_0 \notin (-\infty, 0)$, by

$$g_\alpha^{c,\ell}(\tau_0) := \frac{1}{2\pi i}\sum_\pm \int_{I_\pm^\ell} \frac{1+\tau\tau_0}{\tau - \tau_0} f_\alpha^c(\tau)(1+\tau^2)^{v-1/2}\,d\tau, \tag{29}$$

where τ_0 is to the right of the contours. If we take τ_0 to the left of the contours, with $|\tau_0| > 1$, $\tau_0 \notin (\infty, 0)$, $\operatorname{Re}\tau_0 < 0$, then

$$g_\alpha^{c,\ell}(\tau_0) = (1+\tau_0^2)^{v+1/2} f_\alpha^c(\tau_0) + \frac{1}{2\pi i}\sum_\pm \int_{I_\pm^\ell} \frac{1+\tau\tau_0}{\tau - \tau_0} f_\alpha^c(\tau)(1+\tau^2)^{v-1/2}\,d\tau. \tag{30}$$

Thus we see that $g_\alpha^{c,\ell}$ represents a hyperfunction α_+^c on $(1,-1)$ (cyclic order), with restriction 0 on $(1,\infty)$, and restriction equal to α^c on $(\infty,-1)$.

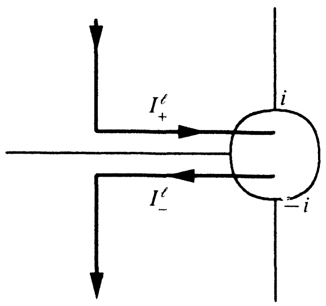


Figure 5. The contours I_+^ℓ and I_-^ℓ used in the construction of α_+^c.

By fixing $I_\pm^\ell$ suitably, we see that $g_\alpha^{c,\ell}(\tau_0)=\mathrm{O}(1)$ as $|\tau_0|\to\infty$, $\operatorname{Re}\tau_0\geqq -2$. By another choice of the contours, and with use of (30), we obtain $g_\alpha^{c,\ell}(\tau_0)=(1+\tau_0^2)^{\nu+1/2}f_\alpha^c(\tau_0)+\mathrm{O}(1)$ as $|\tau_0|\to\infty$, $\tau_0\notin(\infty,-1)$, $\operatorname{Re}\tau_0\leqq -1$.

The invariance of α^c under Γ^∞ implies that $\pi_\nu\begin{pmatrix}1&1\\0&1\end{pmatrix}\alpha_+^c-\alpha_+^c$, considered as a hyperfunction on $(2,-1)$, has support contained in $\{\infty\}$. On a neighborhood of ∞, this difference is represented by

$$h(\tau_0):=\left(\frac{1+\tau_0^2}{1+(\tau_0-1)^2}\right)^{\nu+1/2} g_\alpha^{c,\ell}(\tau_0-1)-g_\alpha^{c,\ell}(\tau_0)\,.$$

In (30) we see that h is holomorphic on $(\infty,-1)$. The estimates of $g_\alpha^{c,\ell}$ imply that h is bounded on a neighborhood of ∞, hence h is holomorphic at ∞ as well. This shows that α_+^c is invariant under Γ^∞.

To finish the proof, we define $\alpha_-^c=\alpha^c-\alpha_+^c$ on $(1,-1)$. □

Remark. We can define a representative $g_\alpha^{c,r}$ of α_-^c in the same way as in (29), but with the other halves of the contours $I_\pm$. In that way we have $g_\alpha^c=g_\alpha^{c,\ell}+g_\alpha^{c,r}$.

5.2. Existence of Γ-decompositions.

Proof of Theorem 2.4. Let q be a Γ-decomposition on $X\subset T$ of an automorphic hyperfunction $\alpha\in A_{-\omega}^\nu(\Gamma)$, as defined in Subsection 2.4. This determines a parting $\alpha=\alpha_{\xi,+}+\alpha_{\xi,-}$ at each $\xi\in X$, by defining $\alpha_{+,\xi}$ as the germ at ξ of $\alpha[\xi,\eta]$ for some $\eta\in X$, $\eta\neq\xi$. Condition (c) in 2.4 ensures that these are Γ-partings, and moreover, that $\alpha_{\gamma\cdot\xi,\pm}=\pi_\nu(\gamma)\alpha_{\xi,\pm}$ for each $\gamma\in\Gamma$. Conversely, if we have such a collection of Γ-partings, then it determines a Γ-decomposition.

Actually, the whole collection of partings, and hence the Γ-decomposition is well determined, if we know the Γ-partings $\alpha=\alpha_{\xi,+}+\alpha_{\xi,-}$ at each ξ in a system of representatives of $\Gamma\backslash X$.

The assumptions of Theorem 2.4 exclude hyperbolic fixed points. So the theorem follows from Proposition 5.4. □

5.3. Geodesic decomposition.

Proposition 5.7. *Let $X \subset \mathscr{C}(\Gamma)$ be a non-empty Γ-invariant set. Suppose that $\alpha \in A^{v}_{-\omega}(\Gamma)$ is cuspidal at each $\xi \in X$. There is a Γ-decomposition* gd *of α on X given by*

$$\langle \varphi, \alpha[\xi, \eta]_{\mathrm{gd}} \rangle = \int_{Q(\xi,\eta)} \varphi(\tau) g(\tau) \frac{d\tau}{\pi(1+\tau^2)} \tag{31}$$

for all $\varphi \in M_\omega^{-v}$, and $\xi, \eta \in X$. The function $g \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ represents α. The contour $Q(\xi, \eta)$ consists of $Q_+(\xi, \eta) \subset \mathfrak{H}^+$ and $Q_-(\xi, \eta) \subset \mathfrak{H}^-$, as indicated in Figure 6. The pieces of $Q_\pm(\xi, \eta)$ that approach ξ, respectively η, are pieces of geodesics in $\mathfrak{H}^\pm$. The definition does not depend on the choice of $Q(\xi, \eta) \subset \operatorname{Supp}(\varphi)$.

We call gd *the* geodesic decomposition *of α.*

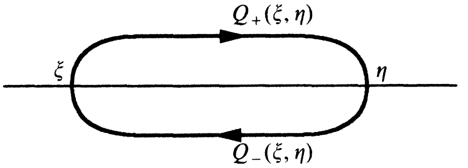


Figure 6. The contour $Q(\xi, \eta)$ used in Proposition 5.7 to define the geodesic decomposition. It is the union of $Q_+(\xi, \eta)$ and $Q_-(\xi, \eta)$. Near ξ and η the contours are pieces of geodesics in $\mathfrak{H}^+$ and $\mathfrak{H}^-$.

Proof. Let $\xi \in X$. Put $\beta = \pi_v(g_\xi^{-1})\alpha$. In (24) we see that the representative g_β^c of $\beta^c = \beta$ satisfies $g_\alpha^c(\tau) = \mathrm{O}(1)$ as $|\operatorname{Im} \tau| \to \infty$, uniformly for $\operatorname{Re} \tau$ in compact sets. Any representative g of α is equivalent to $\pi_v(g_\xi) g_\beta^c$. We conclude that $g(g_\xi \cdot \tau) = \mathrm{O}(1)$ as $|\operatorname{Im} \tau| \to \infty$, uniformly for $\operatorname{Re} \tau$ in compact sets, for each $\xi \in X$.

This shows that the integral over $Q(\xi, \eta)$ is well defined, and does not depend on the geodesics in $\mathfrak{H}^\pm$ along which the points ξ and η are approached. Condition (b) in the definition of Γ-decompositions in Subsection 2.4 is satisfied. A representative of $\alpha[\xi, \eta]_{\mathrm{gd}}$ is given by

$$\tau_0 \mapsto \int_{Q(\xi,\eta)} \frac{-i}{2} \frac{1+\tau\tau_0}{\tau-\tau_0} g(\tau) \frac{d\tau}{\pi(1+\tau^2)},$$

with τ_0 outside $Q(\xi, \eta)$. This representative is holomorphic on $\mathbb{P}^1_{\mathbb{C}} \setminus [\xi, \eta]$.

To check condition (c), we use $\gamma \cdot Q(\xi, \eta)$ as $Q(\gamma \cdot \xi, \gamma \cdot \eta)$. A computation gives the desired result. □

Proposition 5.8. *Let $k \in 2\mathbb{Z}_{>0}$. Suppose that f is a holomorphic cusp form for Γ of weight k. Let $\alpha_f \in A_{-\omega}^{(k-1)/2}(\Gamma)$ be the automorphic hyperfunction corresponding to f, as indicated in* (4). *For all $\xi, \eta \in \mathscr{C}(\Gamma)$:*

$$(\alpha_f[\xi,\eta]_{\mathrm{gd}})^{\langle k\rangle} = \frac{1}{4\pi} \int_{Q_+(\xi,\eta)} f(\tau)(\tau - X)^{k-2}\, d\tau = \frac{1}{4\pi} R_f(\xi,\eta)\,.$$

Remark. The period polynomial R_f has been introduced in (12). See Subsection 4.1 for the intertwining operator $\beta \mapsto \beta^{\langle k\rangle}$.

Proof. A straightforward computation. □

6. Hyperfunction cohomology

6.1. Universal covering group.

Description. The universal covering group $\tilde{G}$ is a central extension of $G = \mathrm{PSL}_2(\mathbb{R})$ with center $\tilde{Z} \cong \mathbb{Z}$. As an analytic variety, it is isomorphic to $\mathfrak{H}^+ \times \mathbb{R}$. This isomorphism is written as $(z, \vartheta) \mapsto \tilde{p}(z)\tilde{k}(\vartheta)$ which covers the isomorphism

$$\mathfrak{H}^+ \times (\mathbb{R} \bmod \pi\mathbb{Z}) \to G : (z, \vartheta) \mapsto p(z)k(\vartheta)\,.$$

There are injective continuous group homomorphisms $\mathbb{R} \to G : x \mapsto \tilde{n}(x) := \tilde{p}(i + x)$, $\mathbb{R}^*_{>0} \to G : y \mapsto \tilde{a}(y) := \tilde{p}(iy)$, and $\mathbb{R} \to G : \vartheta \mapsto \tilde{k}(\vartheta)$, covering respectively $n : \mathbb{R} \to G$, $a : \mathbb{R}_{>0} \to G$, $k : \mathbb{R} \bmod \pi\mathbb{Z} \to G$. The center of $\tilde{G}$ is $\tilde{Z} := \tilde{k}(\pi\mathbb{Z})$. The group $\tilde{P} := \tilde{p}(\mathfrak{H}^+)$ is isomorphic to P; it is the connected component of 1 in the parabolic subgroup $\tilde{Z}\tilde{P}$ of $\tilde{G}$.

By $g \mapsto \hat{g}$ we denote the surjective group homomorphism $\tilde{G} \to G$. We define a lifting $\mathrm{SL}_2(\mathbb{R}) \to \tilde{G} : g \mapsto \tilde{g}$ by $\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\tilde{p}(z) = \tilde{p}\left(\frac{az+b}{cz+d}\right)\tilde{k}(-\arg(cz+d))$, with $-\pi < \arg \leqq \pi$. Some properties: $\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{-1} = \widetilde{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$ if $\arg(ci+d) \in (-\pi, \pi)$, $\widetilde{\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}} = \tilde{p}(z)$, and $\widetilde{\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}} = \tilde{k}(\vartheta)$ for $-\pi < \vartheta < \pi$.

$\tilde{T} := \tilde{P}\backslash\tilde{G}$ is isomorphic to the real line. It is a covering of $T = P\backslash G \cong \mathbb{R} \bmod \pi\mathbb{Z}$; the projection map is $\tilde{P}\tilde{k}(\vartheta) \mapsto \cot\vartheta$. We shall identify $\tilde{T}$ with $\mathbb{R}$, by $\tilde{P}\tilde{k}(\vartheta) \mapsto \vartheta$, and we shall write the action of $g \in \tilde{G}$ as $\vartheta \mapsto \vartheta \cdot g$. So $\vartheta \cdot \tilde{k}(\zeta) = \vartheta + \zeta$. If $\arg(ci+d) \in (-\pi, \pi)$, then $\vartheta \mapsto \vartheta \cdot \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ is the strictly increasing analytic function given by

$$\vartheta \mapsto \arg\big((ia - b)\sin\vartheta + (d - ic)\cos\vartheta\big)$$

on a neighborhood of $\vartheta = 0$. We have $(\vartheta + \pi) \cdot g = \vartheta \cdot g + \pi$ for all $g \in \tilde{G}$.

Discrete subgroup. We define $\tilde{\Gamma}$ to be the full original in $\tilde{G}$ of the discrete subgroup Γ. So $\tilde{\Gamma}$ is a discrete subgroup of $\tilde{G}$ containing $\tilde{Z}$.

The full original $\tilde{\Gamma}_{\text{mod}}$ of the modular group is generated by $w := \tilde{k}(\pi/2) = \widetilde{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$ and $n := \tilde{n}(1) = \widetilde{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$, subject to the relations $w^2 n = n w^2$ and $nwnwn = w^3$ (see, e.g., the discussion in [1], § 13.1).

Action in weight r. Let $r \in \mathbb{C}$. A function f on $\tilde{G}$ has *weight* r if it satisfies $f(g\tilde{k}(\vartheta)) = f(g)e^{ir\vartheta}$ for all $\vartheta \in \mathbb{R}$. A function of weight r is fully determined by the corresponding function $z \mapsto y^{-r/2} f(\tilde{p}(z))$ on $\mathfrak{H}^+$.

The left translation $L_{g_1} f(g) = f(g_1 g)$ in $C^\infty(\tilde{G})$ is a right $\tilde{G}$-action. On functions of weight r, it corresponds to the right $\tilde{G}$-action $g_1 : F \mapsto F|_r g_1$ in the functions on $\mathfrak{H}^+$ defined by $F|_r \tilde{k}(\pi m) = e^{\pi i r m} F$ for $m \in \mathbb{Z}$, and

$$(32) \qquad F|_r \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z) := (cz + d)^{-r} F\left(\frac{az + b}{cz + d}\right)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $-\pi < \arg(ci + d) < \pi$. If the weight r is even, then the center $\tilde{Z}$ acts trivially, and we recover the action of G in weight r.

Formula (32) defining $F|_r \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ makes sense for functions on $\mathfrak{H}^-$ as well. It turns out that we obtain a right action of $\tilde{G}$ on the functions on $\mathfrak{H}^+ \cup \mathfrak{H}^-$, if we put

$$(33) \qquad F|_r \tilde{k}(\pi m) := e^{\pm \pi i r m} F \quad \text{for } z \in \mathfrak{H}^\pm \text{ and } m \in \mathbb{Z}\,.$$

Action in the holomorphic functions on $\mathfrak{H}^+ \cup \mathfrak{H}^-$. Let $\mathcal{H}_r := \mathcal{O}(\mathfrak{H}^+ \cup \mathfrak{H}^-)$, with the right action of $\tilde{G}$ of weight $r \in \mathbb{C}$. For each $h \in \mathcal{H}_r$, let $I(h)$ be the set of points of $T \subset \mathbb{P}^1_{\mathbb{C}}$ to which h has a holomorphic extension.

Property (1) of the period functions can be rewritten as $\psi = \psi|_{2s} \widetilde{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} + \psi|_{2s} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Note that $\psi \in \mathcal{H}_{2s}$ and $I(\psi) \supset (0, \infty)$.

6.2. Restricted cohomology. Let Δ be a group. In this paper we consider group cohomology with values in *right* Δ-modules.

Resolutions. Let $X \neq \emptyset$ be a set on which Δ acts from the right. We define a complex of $\mathbb{Z}\Delta$-modules $\mathcal{Y}.(X)$ by $\mathcal{Y}_n(X) := \mathbb{Z} \cdot X^{n+1}$, the boundary map determined by

$$\partial(\xi_0, \ldots, \xi_n) := \sum_{l=0}^{n} (-1)^{l+1} (\xi_0, \ldots, \hat{\xi}_l, \ldots, \xi_n)\,,$$

and the Δ-action by $(\xi_0, \ldots, \xi_n)\delta := (\xi_0 \cdot \delta, \ldots, \xi_n \cdot \delta)$ for $\delta \in \Delta$. This is a complex of free $\mathbb{Z}\Delta$-modules if Δ acts freely on X.

Cohomology. Let A be a right Δ-module. We define $H^{\cdot}_X(\Delta, A)$ as the cohomology of the complex $\mathrm{Hom}_\Delta(\mathcal{Y}.(X), A)$.

In the special case $\Delta = \Gamma$ and $X = \mathscr{C}(\Gamma)$, the group $H^i_X(\Gamma, A)$ is usually called a parabolic cohomology group. If $\tilde{X} \subset \tilde{T}$ is the full original of $\mathscr{C}(\Gamma)$, then we call also $H^i_{\mathrm{par}}(\tilde{\Gamma}, A) := H^i_{\tilde{X}}(\tilde{\Gamma}, A)$ a *parabolic cohomology group.*

1-*cocycles.* We are mainly interested in $H^1_X(\Delta, A)$. Its elements are represented by homogeneous 1-cocycles $c : X^2 \to A$, satisfying $c(\xi_0, \xi_1) + c(\xi_1, \xi_2) = c(\xi_0, \xi_2)$, $c(\xi, \xi) = 0$, and $c(\xi_0, \xi_1)\delta = c(\xi_0\delta, \xi_1\delta)$. This cocycle is a coboundary, if there is a Δ-equivariant $b : X \to A$ such that $c(\xi_0, \xi_1) = b(\xi_1) - b(\xi_0)$.

In the case that X consists of one Δ-orbit on which Δ acts freely, the choice of a base point identifies X with Δ. Thus we obtain the usual group cohomology $H^1(\Delta, A)$, described by inhomogeneous cocycles $\eta : \Gamma \to A$, satisfying $\eta(\gamma\delta) = \eta(\gamma)\delta + \eta(\delta)$, modulo inhomogeneous coboundaries $\delta \mapsto a\delta - a$ with $a \in A$. An inhomogeneous cocycle corresponds to a homogeneous one by $c(\gamma_0, \gamma_1) = \eta(\gamma_1\gamma_0^{-1})\gamma_0$.

Definition 6.1. Let Δ_1 be a subgroup of Δ. An inhomogeneous 1-cocycle $\eta : \Delta \to A$ is Δ_1-*trivial* if there exists an $a \in A$ such that $\eta(\gamma) = a(\gamma - 1)$ for all $\gamma \in \Delta_1$.

Lemma 6.2. *Let X be as above. The group $H^1_X(\Delta, A)$ is canonically isomorphic to the subgroup of those classes in $H^1(\Delta, A)$ which are represented by a cocycle that is Δ^ξ-trivial for all $\xi \in X$.*

Remarks. So, if Δ acts freely on X, then $H^1_X(\Delta, A)$ is canonically isomorphic to $H^1(\Delta, A)$.

If $\mathscr{C}(\Gamma)$ consists of one Γ-orbit, then the parabolic cohomology group $H^1_{\mathscr{C}(\gamma)}(\Gamma, A)$ has a description similar to the definition of the first parabolic cohomology group in [9], § 8.1.

Proof. Fix $\xi \in X$. The map $(\gamma_0, \ldots, \gamma_n) \mapsto (\xi\gamma_0, \ldots, \xi\gamma_n)$ induces a chain map from the standard resolution $F.$ of $\mathbb{Z}\Delta$ (where $F_n = \mathbb{Z}\Delta^{n+1}$) to $\mathscr{Y}.(X)$. Different $\xi \in X$ lead to homotopic maps. This induces a canonical map $f^j : H^j_X(\Delta, A) \to H^j(\Delta, A)$ for all $j \geqq 0$.

The map f^1 is injective. Indeed, to a 1-cocycle c on X^2 is associated the homogeneous cocycle $c_1(\gamma_0, \gamma_1) = c(\xi\gamma_0, \xi\gamma_1)$. Suppose that $c_1(\gamma_0, \gamma_1) = a(\gamma_1 - \gamma_0)$ with $a \in A$. Let Ξ be a set of representatives of X/Δ. Define $b(\vartheta\delta) = \big(a + c(\xi, \vartheta)\big)\delta$ for $\vartheta \in \Xi$ and $\delta \in \Delta$; then $db = c$.

The cocycles c_1 obtained in this way are Δ^ϑ-trivial for each $\vartheta \in X$, as one sees from $c_1(1, \gamma) = c(\xi, \xi\gamma) = c(\xi, \vartheta) + c(\vartheta, \vartheta\gamma) + c(\vartheta\gamma, \xi\gamma) = c(\vartheta, \xi)(\gamma - 1) + 0$ if $\gamma \in \Delta^\vartheta$.

On the other hand, suppose that for each $\vartheta \in X$ there exist $a_\vartheta \in A$ such that $\eta(\gamma) = a_\vartheta(\gamma - 1)$ for all $\gamma \in \Delta^\vartheta$. We can arrange that $a_{\vartheta\delta} = a_\vartheta\delta - \eta(\delta)$ for all $\vartheta \in X, \delta \in \Delta$. Then $c(\vartheta, \zeta) := a_\vartheta - a_\zeta$ is a homogeneous cocycle on X^2 such that $c_1(1, \delta) = \eta(\delta) - a_\xi(\delta - 1)$ for all $\delta \in \Delta$. □

Inclusions. Lemma 6.2 allows us to identify $H^1_X(\Delta, A)$ with its image in $H^1(\Delta, A)$. If Δ acts on $X_1 \supset X$ as well, then the natural map $H^1_{X_1}(\Delta, A) \to H^1_X(\Delta, A)$ is injective.

Parabolic cohomology for the modular group. Consider the group $\tilde{\Gamma}_{\mathrm{mod}}$ acting on the full original $\tilde{X}$ of $\mathscr{C}(\Gamma_{\mathrm{mod}})$ in $\tilde{T}$. A class in $H^1_{\mathrm{par}}(\tilde{\Gamma}_{\mathrm{mod}}, A)$ can be represented by an inhomogeneous cocycle η that vanishes on the generator $n = \widetilde{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$, and is determined by its value p on the other generator $w^{-1} = \widetilde{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$, satisfying $p(w^{-1}+1)(n-1) = 0$ and $p(nw^{-1}n + n) = p$ (see the relations in Subsection 6.1). The class is trivial if $p = a(w^{-1} - 1)$ for some $a \in A$ satisfying $a(n-1) = 0$. The relation $p(w^{-1}+1)(n-1) = 0$ follows from the other relation:

$$\begin{aligned}(nw^{-1}n + n - 1)(1 - w^{-1}n) &= nw^{-1}n + n - 1 - nw^{-1}nw^{-1}n - nw^{-1}n + w^{-1}n \\ &= (w^{-1}+1)(n-1)\,.\end{aligned}$$

(This computation was shown to me by J. Lewis, for the module $\mathscr{H}_r$.) So for each right $\tilde{\Gamma}_{\mathrm{mod}}$-module:

$$(34) \quad H^1_{\mathrm{par}}(\tilde{\Gamma}_{\mathrm{mod}}, A) \cong \{p \in A : p(nw^{-1}n + n) = p\} \bmod \{a(w^{-1} - 1) : a \in A, an = a\}\,.$$

Consider the right $\tilde{\Gamma}_{\mathrm{mod}}$-module $\mathscr{H}_{2s}$, with the action of weight $2s$. We take for $\tilde{X} \subset \tilde{T}$ the full original of $\mathscr{C}(\Gamma_{\mathrm{mod}})$. The functional equation (1) for the period functions is the relation $\psi|_{2s}(nw^{-1}n + n) = \psi$.

Let $2s \notin 2\mathbb{Z}$. Take $a = \psi|_{2s}(w^{-1}+1)(w^{-2}-1)^{-1}$. Note that $w^{-2} - 1$ is invertible if $2s \notin 2\mathbb{Z}$. Thus, we see that $H^1_{\tilde{X}}(\tilde{\Gamma}_{\mathrm{mod}}, \mathscr{H}_{2s}) = 0$. We might conclude that (1) is a triviality. But that does not take into account the fact that the period functions ψ satisfy $I(\psi) \supset (0, \infty)$, and that property is not clear for a general element of $a|_{2s}(w^{-1} - 1)$, $a \in \mathscr{H}_{2s}$. This aspect is ignored, if we force (1) into the framework of group cohomology with values in $\mathscr{H}_{2s}$.

If $p = \psi_\alpha$ in the situation of Proposition 2.3, then $a = \psi|_{1+2\nu}(w^{-1}+1)(w^{-2}-1)^{-1}$ corresponds to $-f_\alpha$.

6.3. Cohomology with values in the compactly supported hyperfunctions on the line.

Principal series representations. Usually one induces representations from the parabolic subgroup $\tilde{Z}\tilde{P}$ to $\tilde{G}$. Here we induce from $\tilde{P}$ to $\tilde{G}$.

Let $(\tilde{\pi}_\nu, \tilde{M}^\nu)$ be the representation of $\tilde{G}$ in the functions on $\tilde{G}$ that transform on the left according to $\tilde{p}(z) \mapsto y^{1/2+\nu}$. This representation can be realized in the functions on $\tilde{T}$. This leads to the action $\tilde{\pi}_\nu(\tilde{G})$ in the analytic functions $\tilde{M}^\nu_\omega$ on $\tilde{T}$, and in the space of hyperfunctions $\tilde{M}^\nu_{-\omega} := \mathscr{B}(\mathbb{R})$. In the coordinate ϑ the action of $\tilde{\pi}_\nu(\tilde{k}(\zeta))$ is the translation over ζ.

Let $\tilde{M}^\nu_{-\omega,c}$ be the subspace of $\tilde{M}^\nu_{-\omega}$ of hyperfunctions with bounded support. The space $\tilde{M}^\nu_{-\omega,c}$ is invariant under $\tilde{\pi}_\nu$, and there is a duality between $(\tilde{\pi}_\nu, \tilde{M}^\nu_\omega)$ and $(\tilde{\pi}_{-\nu}, \tilde{M}^{-\nu}_{-\omega,c})$, that can be described with explicit integrals, see Subsection 3.1.

Identifications.

$$(35)\qquad M^{\nu}_{\omega} = (\tilde{M}^{\nu}_{\omega})^{\tilde{Z}}, \quad M^{\nu}_{-\omega} = (\tilde{M}^{\nu}_{-\omega})^{\tilde{Z}}, \quad A^{\nu}_{-\omega}(\Gamma) = (\tilde{M}^{\nu}_{-\omega})^{\tilde{\Gamma}}.$$

The action of $\tilde{\pi}_\nu(g)$ in the $\tilde{\pi}_\nu(\tilde{Z})$-invariant spaces on the right of the equality signs corresponds to the action of $\pi_\nu(\hat{g})$ in the spaces on the left.

If $\beta \in \tilde{M}^{\nu}_{-\omega,c}$, then $\sum_{\zeta\in\tilde{Z}} \tilde{\pi}_\nu(\zeta)\beta$ is well defined, and determines an element of $(\tilde{M}^{\nu}_{-\omega})^{\tilde{Z}}$. We denote the corresponding element in $M^{\nu}_{-\omega}$ by $\sigma\beta$.

Right module. In the cohomology group $H^{\cdot}_{\tilde{X}}(\tilde{\Gamma}, \tilde{M}^{\nu}_{-\omega,c})$, we consider $\tilde{M}^{\nu}_{-\omega,c}$ as a right $\tilde{G}$-module by $\alpha g = \tilde{\pi}_\nu(g^{-1})\alpha$.

Proof of Proposition 2.5. Given α and q we construct $c_{\alpha,q}$ in the following way:

(a) Let $\xi, \eta \in \tilde{X}$, $\xi < \eta < \xi + \pi$. Take an open interval $\tilde{I} \supset [\xi, \eta]$ of length less than π. Under $\vartheta \mapsto \tau = \cot\vartheta$, this is isomorphic to an open interval $I \subset T$, $I \neq T$, such that $I \supset [\cot\eta, \cot\xi]$. We define $c_{\alpha,q}(\xi,\eta)$ as the hyperfunction on $\tilde{I}$ that corresponds to $\alpha[\cot\eta, \cot\xi]_q$ on I, and we extend $c_{\alpha,q}(\xi,\eta)$ to $\mathbb{R} \cong \tilde{T}$ by zero outside I.

The properties of $\alpha[\cdot,\cdot]_q$ imply $c_{\alpha,q}(\xi,\vartheta) + c_{\alpha,q}(\vartheta,\eta) = c_{\alpha,q}(\xi,\eta)$ if $\xi < \vartheta < \eta < \xi + \pi$, and $c_{\alpha,q}(\xi\cdot\gamma, \eta\cdot\gamma) = \tilde{\pi}_\nu(\gamma)^{-1}\big(c_{\alpha,q}(\xi,\eta)\big) = c_{\alpha,q}(\xi,\eta)\gamma$ for all $\gamma \in \tilde{\Gamma}$. We use the fact that $\cot(\vartheta\cdot g) = \hat{g}^{-1}\cdot\cot\vartheta$, where $g \mapsto \hat{g}$ is the projection $\tilde{G} \to G$.

(b) If $\eta \geqq \xi + \pi$, then we take intermediate points $\vartheta_0 = \xi < \vartheta_1 < \cdots < \vartheta_k = \eta$, with $\vartheta_j < \vartheta_{j-1} + \pi$, and define $c_{\alpha,q}(\xi,\eta) := \sum_{j=1}^{k} c_{\alpha,q}(\vartheta_{j-1}, \vartheta_j)$. This does not depend on the choice of the intermediate points.

(c) $c_{\alpha,q}(\xi,\xi) := 0$, $c_{\alpha,q}(\xi,\eta) := -c_{\alpha,q}(\eta,\xi)$ if $\eta < \xi$.

This defines a 1-cocycle whose support has the property indicated in the proposition. If $\xi < \eta$ in $\tilde{X}$, then $c_q(\xi,\eta)$ and the $\tilde{\Gamma}$-invariant hyperfunction on $\tilde{T}$ corresponding to α have the same restriction to the open interval (ξ,η).

Lemma 6.4 below shows that the cohomology class of $c_{\alpha,q}$ does not depend on q. Lemma 6.5 gives the injectivity of the map from $A^{\nu}_{-\omega}(\Gamma, X)$ into the cohomology group. Lemma 6.3 characterizes the image. □

Lemma 6.3. *For each* 1*-cocycle* $c \in \mathrm{Hom}_{\tilde{\Gamma}}(\mathscr{Y}_1(\tilde{X}), \tilde{M}^{\nu}_{-\omega,c})$ *that satisfies*

$$\mathrm{Supp}\big(c(\xi,\eta)\big) \subset [\xi,\eta] \quad \textit{for all } \xi,\eta \in \tilde{X},\ \xi < \eta,$$

there is a unique $\alpha \in A^{\nu}_{-\omega}(\Gamma, X)$ *and a unique* Γ*-decomposition* p *of* α *on* X *such that* $c = c_{\alpha,p}$.

Proof. Let c be given. For $x, y \in T$, $x \neq y$, we can find $\xi, \eta \in \tilde{X}$ such that $x = \cot\xi$, $y = \cot\eta$, and $\eta < \xi < \eta + \pi$. We define $A(x,y) := \sigma c(\xi,\eta) \in M^{\nu}_{-\omega}$. So $\mathrm{Supp}\big(A(x,y)\big) \subset [x,y]$. The freedom in the choice of ξ and η is a translation over a multiple of π, and $\tilde{k}(\pi) \in \tilde{Z} \subset \tilde{\Gamma}$, hence the choice of ξ and η does not influence the definition of $A(x, y)$. If there are α and q with $c_{\alpha,q} = c$, then $\alpha[x, y]_q = A(x, y)$. The properties $A(x, z) + A(z, y) = A(x, y)$ and $A(\gamma \cdot x, \gamma \cdot y) = \pi_v(\gamma) A(x, y)$ are easily checked.

Consider $x, y, z, u \in T, x \neq y, z \neq u$. Choose $\xi, \eta, \zeta, v \in \tilde{T}$ above these elements such that $\eta < \xi < \eta + \pi$ and $v < \zeta < v + \pi$. Put

$$p_1 := c(\xi, \eta) + c(\eta, \xi - \pi) \quad \text{and} \quad p_2 := c(\zeta, v) + c(v, \zeta - \pi).$$

Then $A(x, y) + A(y, x) = \sigma p_1$, and $A(z, u) + A(u, z) = \sigma p_2$. The cocycle properties show that $\sigma(p_2 - p_1) = \sigma\big(c(\xi, \zeta) + c(\zeta - \pi, \xi - \pi)\big) = 0$. This implies that $\alpha := A(x, y) + A(y, x)$ does not depend on the choice of x and y. We have $\alpha \in A^{v}_{-\omega}(\Gamma)$ and $\alpha[x, y]_q := A(x, y)$ defines a Γ-decomposition of α such that $c_{\alpha,q} = c$. □

Lemma 6.4. *Let $\alpha \in A^{v}_{-\omega}(\Gamma, X)$, and let p be a Γ-decomposition of α on X. Then each Γ-decomposition q of α on X has the form $q = p\langle h \rangle$, where*

$$\alpha[\cot \xi, \cot \eta]_{p\langle h \rangle} := \alpha[\cot \xi, \cot \eta]_p + \sigma h(\eta) - \sigma h(\xi),$$

for $\eta \geqq \xi < \eta + \pi$, $\xi, \eta \in \tilde{X}$, and where $h \in \mathrm{Hom}_{\tilde{\Gamma}}\big(\mathscr{Y}_0(\tilde{X}), \tilde{M}^{v}_{-\omega,c}\big)$ satisfies $\mathrm{Supp}\big(h(\xi)\big) \subset \{\xi\}$ for $\xi \in \tilde{X}$.

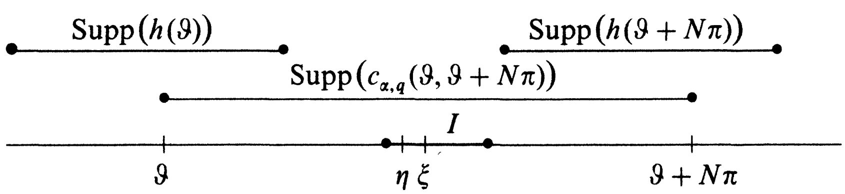
For such h we have $c_{\alpha,p\langle h \rangle} = c_{\alpha,p} + dh$.

Proof. The correspondence in Lemma 6.3 associates α and $p\langle h \rangle$ to the cocycle $c_{\alpha,p} + dh$. This shows that $p\langle h \rangle$ is a Γ-decomposition.

Let p and q be Γ-decompositions of α on X. The cocycle $c := c_{\alpha,p} - c_{\alpha,p}$ satisfies $\mathrm{Supp}\big(c(\xi, \eta)\big) \subset \{\xi, \eta\}$. For $\xi \neq \eta$, we have $c(\xi, \eta) = c_l(\xi, \eta) + c_r(\xi, \eta)$ with $\mathrm{Supp}\big(c_l(\xi, \eta)\big) \subset \{\xi\}$ and $\mathrm{Supp}\big(c_r(\xi, \eta)\big) \subset \{\eta\}$. A consideration of the cocycle relation for three different points shows that $c_l(\xi, \eta) = c_l(\xi)$, $c_r(\xi, \eta) = c_r(\eta)$, and $c_l(\xi) = -c_r(\xi)$. The $\tilde{\Gamma}$-behavior is $c_r(\xi \cdot \gamma) = c_r(\xi)\gamma$. Take $h := c_r$. Then $c_{\alpha,q} = c_{\alpha,p} + dh$, with h satisfying the condition in the lemma. □

Lemma 6.5. *Let $\alpha \in A^{v}_{-\omega}(\Gamma, X)$ and let q be a Γ-decomposition of α on X. If $c_{\alpha,q}$ is a coboundary, then $\alpha = 0$.*

Proof. Suppose $c_{\alpha,q} = dh$ for some $h \in \mathrm{Hom}_{\tilde{\Gamma}}\big(\mathscr{Y}_0(\tilde{X}), \tilde{M}^{v}_{b}\big)$. Fix $\vartheta \in \tilde{X}$. The support of $h(\vartheta)$ is contained in some bounded closed interval J. Take $N \in \mathbb{N}$ large, such that $N\pi + \min(J) > \max(J) + 4\pi$. There is a closed interval I of length 2π between ϑ and $\vartheta + N\pi$ that does not intersect J or $J + N\pi$.



The restriction of $c_{\alpha,p}(\vartheta, \vartheta + N\pi) = h(\vartheta)\tilde{k}(N\pi) - h(\vartheta)$ to I vanishes. So for any $\xi, \eta \in I$, $\eta < \xi < \eta + \pi$, we have $\mathrm{Supp}(c_{\alpha,\pi}(\xi, \eta)) \subset \{\eta, \xi\}$, and hence

$$\mathrm{Supp}\, \alpha[\cot \xi, \cot \eta]_p \subset \{\cot \xi, \cot \eta\}.$$

This implies that α has restriction zero on each open interval in T bounded by two different points of X. As X is infinite, we have $\alpha = 0$. □

Proof of Theorem 2.6. If $\nu \in \frac{1}{2} + \mathbb{Z}_{\geqq 0}$, let X be the Γ-orbit of a point $\xi \in T$ with $\Gamma^{\xi} = \{1\}$, and let $X = \mathscr{C}(\Gamma)$ otherwise. Then $A^{\nu}_{-\omega}(\Gamma) = A^{\nu}_{-\omega}(\Gamma, X)$ by Theorem 2.4. Let $\tilde{X}$ be the full original of X in $\tilde{T}$. Proposition 2.5 gives an injection $A^{\nu}_{-\omega}(\Gamma) \to H^1_{\tilde{X}}(\tilde{\Gamma}, \tilde{M}^{\nu}_{-\omega,c})$. Lemma 6.2 identifies $H^1_{\tilde{X}}(\tilde{\Gamma}, \tilde{M}^{\nu}_{-\omega,c})$ with $H^1(\tilde{\Gamma}, \tilde{M}^{\nu}_{-\omega,c})$ if $\nu \in \frac{1}{2} + \mathbb{Z}_{\geqq 0}$, and with $H^1_{\mathrm{par}}(\tilde{\Gamma}, \tilde{M}^{\nu}_{-\omega,c})$ for general values of ν. □

7. Image in the holomorphic functions of weight $1 - 2\nu$

7.1. Construction of the intertwining operator P. This subsection contains the proof of Proposition 2.7. We construct the operator P in a number of steps. To show that P arises naturally, we take the first step in a more general situation.

Proposition 7.1. *Let $\nu \in \mathbb{C}$. For each $r \in \mathbb{C}$ there is a non-trivial intertwining operator ϱ_r from $\tilde{M}^{\nu}_{-\omega,c}$ to the real analytic functions on $\tilde{G}$ of weight r with the action $g \mapsto L_{g^{-1}}$ of $\tilde{G}$ by left translation.*

The functions in $\varrho_r \tilde{M}^{\nu}_{-\omega,c}$ are eigenfunctions of the Casimir operator with eigenvalue $\frac{1}{4} - \nu^2$. Furthermore, $\mathbf{E}^{\pm}(\varrho_r \beta) = (1 - 2\nu \pm r)\varrho_{r\pm 2}\beta$ for each $\beta \in \tilde{M}^{\nu}_{-\omega,c}$.

Proof. The space $\tilde{M}^{-\nu}_{\omega}$ of analytic functions, considered as a subspace of $C^{\infty}(\tilde{G})$, contains a one-dimensional subspace of functions of weight $r \in \mathbb{C}$. This subspace is spanned by the function $\varphi_r : \vartheta \mapsto e^{ir\vartheta}$. If $r \in 2\mathbb{Z}$, then this function corresponds to $\varphi_r : \tau \mapsto \left(\frac{\tau + i}{\tau - i}\right)^{r/2}$ in $M^{\nu}_{\omega} \cong (\tilde{M}^{\nu}_{\omega})^{\tilde{Z}}$

Let $\beta \in \tilde{M}^{\nu}_{-\omega,c}$. We define $(\varrho_r \beta)(g) := \langle \tilde{\pi}_{-\nu}(g)\varphi_r, \beta \rangle$. This defines a real analytic function $\varrho_r \beta$ on $\tilde{G}$ that has weight r. Moreover, $\varrho_r(\tilde{\pi}_{\nu}(x)\beta) = L_{x^{-1}}\varrho_r \beta$ for $x \in \tilde{G}$. The differentiation relations between the ϱ_r are a consequence of $d\tilde{\pi}_{-\nu}(\mathbf{W})\varphi_r = ir\varphi_r$ and $d\tilde{\pi}_{-\nu}(\mathbf{E}^{\pm})\varphi_r = (1 - 2\nu \pm r)\varphi_{r\pm 2}$. □

Intertwining operator to the holomorphic functions on $\mathfrak{H}^+$. Let us take r in such a way that φ_r is a lowest weight vector in a discrete series type representation of $\tilde{G}$, i.e., $r = 1 - 2\nu$. Then $\mathbf{E}^-(\varphi_{1-2\nu}\beta) = 0$ for all $\beta \in \tilde{M}^{-\nu}_{\omega}$. This is equivalent to the corresponding functions $z \mapsto y^{-1/2+\nu}(\varrho_{1-2\nu}\beta)(\tilde{p}(z))$ being holomorphic. Hence

$$\mathsf{P}^+\beta(z) := y^{-1/2+\nu}(\varrho_{1-2\nu}\beta)(\tilde{p}(z)) \tag{36}$$

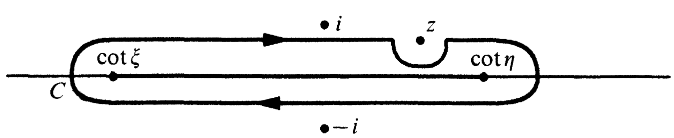
for $z \in \mathfrak{H}^+$ and $\beta \in \tilde{M}^{\nu}_{-\omega,c}$, defines an intertwining operator P^+ from $\tilde{M}^{\nu}_{-\omega,c}$ to the holomorphic functions on $\mathfrak{H}^+$ with the action of weight $1-2\nu$.

Integral representation. If $\mathrm{Supp}(\beta) \subset [\eta, \xi] \subset \tilde{T}$ for some $\beta \in \tilde{M}^{\nu}_{-\omega,c}$, and $\tilde{\chi}$ is holomorphic on a neighborhood of $[\eta, \xi]$, then $\langle \tilde{\chi}, \beta \rangle = \int_{\tilde{C}} \tilde{\chi}(\vartheta)\tilde{g}(\vartheta)\,\frac{d\vartheta}{\pi}$, for some contour encircling $[\eta, \xi]$ once in positive direction, and $\tilde{g}$ a representative of β. We write this as

$$\langle \tilde{\chi}, \beta \rangle = \int_{C} \chi(\tau) g(\tau)\, \frac{d\tau}{\pi(1+\tau^2)}, \tag{37}$$

with C the image of $\tilde{C}$ under $\vartheta \mapsto \cot\vartheta$ with opposite orientation, and χ and g multivalued functions on an open set in $\mathbb{P}^1_{\mathbb{C}}$. In general, this does not help much. But if the interval $[\eta, \xi]$ has length smaller than π, then the functions are single valued.

Let us take $[\eta, \xi] \subset (-\pi, 0]$. The contour C has the form indicated in Figure 7. The interval $[\eta, \xi]$ is sent to $[\cot\xi, \cot\eta]$ (equal to $[\infty, \cot\eta]$ if $\xi = 0$).



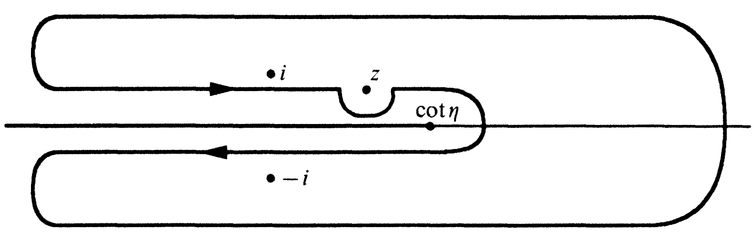


Figure 7. Contours C in the integral representations (37) and (38), with $\mathrm{Supp}(\beta) \subset [\xi, \eta] \subset (-\pi, 0]$. The upper case is valid only for $-\pi < \eta < \xi < 0$, the lower case for $-\pi < \eta < \xi \leqq 0$.

Take $z \in \mathfrak{H}^+$. To compute $\mathrm{P}^+\beta(z) = y^{\nu-1/2}\langle \tilde{\pi}_{-\nu}(\tilde{p}(z))\varphi_{1-2\nu}, \beta\rangle$, we have to take $\tilde{\chi}(\vartheta) = \tilde{\chi}_z(\vartheta)$ equal to $y^{\nu-1/2}\tilde{\pi}_{-\nu}(\tilde{p}(z))\varphi_{1-2\nu}(\vartheta)$ for ϑ in a neighborhood of $[\eta, \xi]$. We have $\tilde{k}(\vartheta)\tilde{p}(z) = \tilde{p}\left(\dfrac{z\cos\vartheta + \sin\vartheta}{-z\sin\vartheta + \cos\vartheta}\right)\tilde{k}(-\arg(\cos\vartheta - z\sin\vartheta))$ for $-\pi < \vartheta < \pi$, hence

$$\begin{aligned}\tilde{\chi}_z(\vartheta) &:= y^{\nu-1/2}\tilde{\pi}_{-\nu}(\tilde{p}(z))\varphi_{1-2\nu}(\vartheta)\\ &= y^{\nu-1/2}\left(\mathrm{Im}\,\frac{z\cos\vartheta + \sin\vartheta}{-z\sin\vartheta + \cos\vartheta}\right)^{1/2-\nu} e^{-i(1-2\nu)\arg(\cos\vartheta - z\sin\vartheta)}.\end{aligned}$$

We write $(\cos\vartheta - z\sin\vartheta)^2$ as

$$\frac{1}{2}\left((1+iz)e^{2i\vartheta} + (1-iz)\right)\cdot\frac{1}{2}\left((1-iz)e^{-2i\vartheta} + (1+iz)\right),$$

to conclude that $\tilde{\chi}_z(\vartheta)$ is equal to

$$2^{1-2\nu}\big((1+iz)e^{2i\vartheta}+(1-iz)\big)^{\nu-1/2}\big((1-iz)e^{-2i\vartheta}+(1+iz)\big)^{\nu-1/2},$$

with the arguments chosen in such a way that the whole expression equals 1 at $\vartheta = 0$. (The argument of the first factor has period π as a function of ϑ; the second factor has a decreasing argument.)

With $\tau = \cot\vartheta$, we have $\dfrac{\tau - z}{\tau \mp i} = \dfrac{1}{2}\big((1 \pm iz)e^{\pm 2i\vartheta} + 1 \mp iz\big)$. Hence

$$\chi_z(\tau) = \left(\frac{\tau - z}{\tau + i}\right)^{\nu - 1/2}\left(\frac{\tau - z}{\tau - i}\right)^{\nu - 1/2}$$

satisfies $\tilde{\chi}_z(\vartheta) = \chi_z(\cot\vartheta)$ for ϑ in a neighborhood of $[\eta, \xi]$. Here we take $\arg\left(\dfrac{\tau - z}{\tau \pm i}\right)$ equal to 0 at $\tau = \infty$, with discontinuities as indicated in Figure 8. Thus we find the integral representation

$$\mathsf{P}^{+}\beta(z) = \int_C \chi_z(\tau)g(\tau)\,\frac{d\tau}{\pi(1+\tau^2)}, \tag{38}$$

if $\operatorname{Supp}(\beta) \subset [\eta, \xi] \subset (-\pi, 0]$, with representative $\vartheta \mapsto g(\cot\vartheta)$, and C as in Figure 7 not intersecting the lines of discontinuity in Figure 8.

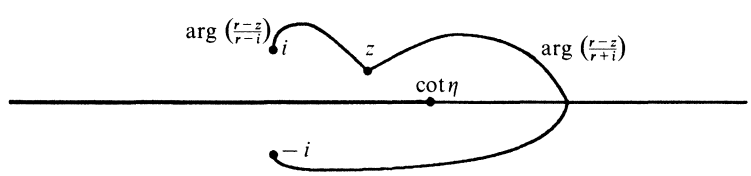


Figure 8. Sets of discontinuity of the argument of $\chi_z(\tau)$.

Definition of P. If $\operatorname{Supp}(\beta) \subset [\eta, \xi] \subset (-\pi, 0]$, then the integral representation (38) makes sense if we move z into the lower half plane passing the real axis at points of $(\cot\eta, \infty)$. We take this as the definition of $\mathsf{P}\beta(z)$ on $\mathbb{C} \setminus (-\infty, \cot\eta]$. (If $\xi < 0$, there is also an extension to $\mathbb{C} \setminus (\cot\xi, \infty)$. On $\mathfrak{H}^-$, this extension is equal to $e^{4\pi i\nu}\mathsf{P}\beta$.)

If both β_1 and β_2 satisfy $\operatorname{Supp}(\beta_j) \subset [\eta, \xi]$, then $\mathsf{P}(\beta_1 + \beta_2) = \mathsf{P}\beta_1 + \mathsf{P}\beta_2$. If $h \in \tilde{G}$ satisfies $\operatorname{Supp}(\beta) \cdot h^{-1} \subset (-\pi, 0]$, then $\mathsf{P}\tilde{\pi}_\nu(h)\beta = \mathsf{P}\beta|_{1-2\nu}h^{-1}$ on $\mathbb{C} \setminus (-\infty, t)$ for some $t \in \mathbb{R}$. (We have this relation already for $z \in \mathfrak{H}^+$.) If the support of $\beta \in \tilde{M}^{\nu}_{-\omega,c}$ is contained in an interval with length strictly smaller than π, we take $g \in \tilde{G}$ such that $\operatorname{Supp}(\tilde{\pi}_\nu(g)\beta) = \operatorname{Supp}(\beta) \cdot g^{-1} \subset (-\pi, 0]$, and put $\mathsf{P}\beta := \big(\mathsf{P}\tilde{\pi}_\nu(g)\beta\big)|_{1-2\nu}g$. This does not depend on the choice of g. We can write any $\beta \in \tilde{M}^{\nu}_{-\omega,c}$ as a sum of hyperfunctions on $\tilde{T}$ with support of length strictly smaller than π. In this way we extend P to all hyperfunctions with bounded support.

This finishes the proof of Proposition 2.7. □

Cocycles. Propositions 2.5 and 2.7 imply that that there is a 1-cocycle $\mathsf{P}c_{\alpha,q}$ on $\tilde{X}$ with values in $\mathscr{H}_{1-2\nu}$ for each $\alpha \in A^{\nu}_{-\omega}(\Gamma, X)$, and each Γ-decomposition q of α on X. ($X \subset T$ is Γ-invariant, and $\tilde{X}$ is its full original in $\tilde{T}$.) If $\xi, \eta \in \tilde{X}$ satisfy $-\pi < \eta < \xi \leqq 0$, then $I(\mathsf{P}c_{\alpha,q}(\eta, \xi)) \supset (\cot \eta, \infty)$.

7.2. Proof of Proposition 2.8. In this subsection, we take $X = \Gamma \cdot \infty$. So $\tilde{X}$ is the orbit of 0 in $\tilde{T}$ under $\tilde{\Gamma}$. We consider $\alpha \in A^{\nu}_{-\omega}(\Gamma)$ that is cuspidal at ∞. So $\alpha \in A^{\nu}_{-\omega}(\Gamma, X)$. We compute $\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi, 0)(z)$ as explicitly as possible.

Intermediate point. The interval $[-\pi, 0]$ is just too long to use (38) directly.

Choose $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c > 0$. Take $\gamma := \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in \tilde{\Gamma}$, $\eta := 0 \cdot \gamma = \arg(d - ic) \in (-\pi, 0)$ and $\eta_1 := \arg(-a - ic) = (-\pi) \cdot \gamma^{-1} \in (-\pi, 0)$. We use

$$c_{\alpha,\mathrm{gd}}(-\pi, 0) = c_{\alpha,\mathrm{gd}}(\eta_1, 0)\gamma + c_{\alpha,\mathrm{gd}}(\eta, 0)\,.$$

Integral representations for $\mathsf{P}c_{\alpha,\mathrm{gd}}(\eta, 0)$ *and* $\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi, \eta)$. Let h be a representative of α. Proposition 5.7 and equation (38) give

$$\mathsf{P}c_{\alpha,\mathrm{gd}}(\eta, 0)(z) = \sum_{\pm} \int_{Q_\pm(\infty, -d/c)} \chi_z(\tau) h(\tau) \frac{d\tau}{\pi(1+\tau^2)}, \tag{39}$$

and a similar integral, with $Q_\pm(\infty, a/c)$, for $\mathsf{P}c_{\alpha,\mathrm{gd}}(\eta_1, 0)(z)$. We use the Γ-invariance of α to transform the latter integral into

$$\mathsf{P}c_{\alpha,\mathrm{gd}}(\eta_1, 0)(z) = \sum_{\pm} \int_{Q(-d/c, \infty)} \left(\frac{(a\tau+b)^2 + (c\tau+d)^2}{1+\tau^2}\right)^{\nu-1/2} \cdot \chi_z\left(\frac{a\tau+b}{c\tau+d}\right) h(\tau) \frac{d\tau}{\pi(1+\tau^2)}\,. \tag{40}$$

The factor $f(z,\tau) := \left(\frac{(a\tau+b)^2 + (c\tau+d)^2}{1+\tau^2}\right)^{\nu-1/2} \chi_z\left(\frac{a\tau+b}{c\tau+d}\right)$ is holomorphic in $\tau \in \mathbb{P}^1_{\mathbb{C}}$, except for a path from i, via $\frac{di-b}{a-ic}$, $\frac{dz-b}{a-cz}$ and $\frac{-di-b}{a+ic}$, to $-i$, crossing the real axis at a point of $(-\infty, -d/c)$. To determine it completely, we note that

$$f(z, \infty) = (a^2 + c^2)^{\nu-1/2} \chi_z(a/c)\,.$$

With the standard choice of the arguments, $\chi_z(\tau) = (z-\tau)^{2\nu-1}(-i-\tau)^{1/2-\nu}(i-\tau)^{1/2-\nu}$ for $\tau \notin (i + [0,\infty)) \cup (-i + [0,\infty)) \cup (z + [0,\infty))$. This gives

$$\chi_z(a/c) = (a^2+c^2)^{1/2-\nu}(cz-a)^{2\nu-1}\,,$$

for $z \in \mathbb{C} \setminus (-\infty, a/c]$.

For $z \in \mathbb{C} \setminus [-d/c, \infty)$, we use the integral representation (40) to compute $\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi, \eta)(z) = \mathsf{P}c_{\alpha,\mathrm{gd}}(\eta_1, 0)|_{1-2\nu}\gamma(z)$:

$$\text{(41)}\quad \mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi,\eta)(z) = (cz+d)^{2\nu-1}\sum_{\pm}\int_{Q(-d/c,\infty)} f\left(\frac{az+b}{cz+d},\tau\right)h(\tau)\,\frac{d\tau}{\pi(1+\tau^2)}.$$

On the path of integration, we have the equality

$$f\left(\frac{az+b}{cz+d},\tau\right) = \left(\frac{\tau-z}{\tau-i}\right)^{\nu-1/2}\left(\frac{\tau-z}{\tau+i}\right)^{\nu-1/2}\left(\frac{-1}{cz+d}\right)^{2\nu-1},$$

but now the discontinuities occur along a path from i via z to $-i$ that crosses the real line to the left of $-d/c$.

Integral representation for $\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi,0)$. Let $z\in\mathfrak{H}^{\pm}$. To combine (39) and (41), we use another function ω_z, which is equal to χ_z on a neighborhood of $(-\infty, d/c]$, and has its discontinuities along the lines $i[1,\infty)$ and $(-i)[1,\infty)$, and along a path from z to $\pm i$ inside $\mathfrak{H}^{\pm}$. Hence

$$\omega_z(\tau) = (z-\tau)^{2\nu-1}(1+\tau^2)^{1/2-\nu},$$

with the standard choice of $\arg(1+\tau^2)\in(-\pi,\pi)$, and $\arg(z-\tau)$ tending to 0 as τ approaches ∞ along the negative real axis. So $\arg(z-\tau)$ is discontinuous along a path from z via $\pm i$ to ∞ inside $\mathfrak{H}^{\pm}$.

If we move the vertical part of the contours $Q_{\pm}(\infty,-d/c)$ far enough to the left, then the integral in (39) does not change if we replace χ_z by ω_z, provided $z\in\mathbb{C}\setminus\mathbb{R}$. The function

$$\tau\mapsto(cz+d)^{2\nu-1}f\left(\frac{az+b}{cz+d},\tau\right) = -e^{\pm 2\pi i\nu}\left(\frac{\tau-z}{\tau-i}\right)^{\nu-1/2}\left(\frac{\tau-z}{\tau+i}\right)^{\nu-1/2}$$

is a multiple of ω_z on a neighborhood of $[-d/c,\infty)$. Let τ tend to ∞ along the positive axis to see that the factor equals 1. We replace (41) by

$$\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi,\eta)(z) = \sum_{\pm}\int_{Q(-d/c,\infty)}\omega_z(\tau)h(\tau)\,\frac{d\tau}{\pi(1+\tau^2)}.$$

We combine the integrals over $Q_{\pm}(\infty,-d/c)$ and $D_{\pm}(-d/c,\infty)$, and obtain contours $I_{\pm}$ as in Figure 3. (We take care to have $z\in\mathfrak{H}^{\pm}$ inside $I_{\pm}$.) For $z\in\mathbb{C}\setminus\mathbb{R}$, we find

$$\text{(42)}\qquad \mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi,0)(z) = \sum_{\pm}\int_{I_{\pm}}\omega_z(\tau)h(\tau)\,\frac{d\tau}{\pi(1+\tau^2)}.$$

Choice of a representative of α. The automorphic hyperfunction α is cuspidal at ∞. We replace h in (42) by g_α^c in (24):

$$\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi,0)(z) = \sum_{\zeta\in\{+,-\}}\int_{I_\zeta}\omega_z(\tau_0)\frac{1}{2\pi i}\sum_{\varepsilon\in\{+,-\}}\int_{J_\varepsilon}\frac{1+\tau\tau_0}{\tau-\tau_0}f_\alpha^c(\tau)(1+\tau^2)^{\nu-1/2}d\tau\,\frac{d\tau_0}{\pi(1+\tau_0^2)},$$

with the contours J_ε of the same type as I_ζ, in such a way that I_ζ is inside J_ζ.

Next we shrink the J_ε such that they are inside the I_ε. This gives a similar double integral, plus terms coming from the residues. The latter yield, for $z \in \mathfrak{H}^\pm$:

$$\sum_{\zeta \in \{+,-\}} \int_{I_\zeta} \omega_z(\tau_0)(1+\tau_0^2)^{\nu-1/2} f_\alpha^c(\tau_0)\,\frac{d\tau_0}{\pi} = \sum_{\zeta \in \{+,-\}} \int_{I_\zeta} (z-\tau_0)^{2\nu-1} f_\alpha^c(\tau_0)\,\frac{d\tau_0}{\pi}.$$

The argument of $z - \tau_0$ follows the conventions given above. The Fourier expansion (23) of f_α^c converges absolutely on the path of integration. Let us look at one term $\zeta A_{\zeta n}(\alpha)\, e^{\zeta 2\pi i n \tau_0}$, with $n \in \mathbb{Z}_{\geqq 1}$. If $z \in \mathfrak{H}^{-\zeta}$, then we move off the path of integration toward $\zeta i\infty$, and we obtain 0 as the value of the integral. We are left with

$$(43)\qquad \pm \sum_{n=1}^{\infty} A_{\pm n}(\alpha) \int_{I_\pm} (z-\tau_0)^{2\nu-1} e^{\pm 2\pi i n \tau_0}\,\frac{d\tau_0}{\pi}$$
$$= \sum_{n=1}^{\infty} A_{\pm n}(\alpha)\, \frac{\mp 2ie^{\pm \pi i \nu}}{\Gamma(1-2\nu)}\, \frac{e^{\pm 2\pi i n z}}{(2\pi i n)^{2\nu}}.$$

In particular, this contribution vanishes if $\nu \in \frac{1}{2}\mathbb{Z}_{\geqq 0}$.

In the remaining sum of double integrals, we interchange the order of integration. In the interior integrals

$$\sum_{\zeta \in \{+,-\}} \int_{I_\zeta} \omega_z(\tau_0)\, \frac{1+\tau\tau_0}{\tau-\tau_0}\, \frac{d\tau_0}{\pi(1+\tau_0^2)},$$

the integrand is $\mathrm{O}(\tau_0^{-2})$ as $|\tau_0| \to \infty$, and there are no singularities between I_+ and I_-. So this sum vanishes.

We conclude that $\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi, 0)(z)$ is given by (43). Here $z \in \mathbb{C} \setminus \mathbb{R}$, and $\nu \in \mathbb{C}$.

Isomorphism. Now we suppose that $\nu \notin \frac{1}{2} + \mathbb{Z}$, and use the isomorphism $\iota(\nu)$ determined by (18). Corollary 4.7 gives a relation between the coefficients $A_n(\alpha)$ and $A_n(\iota(\nu)\alpha)$. From (43) we conclude, for $z \in \mathfrak{H}^\pm$:

$$\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi, 0)(z) = \frac{-2i\sqrt{\pi}\, e^{\pm \pi i \nu}}{\Gamma(1-\nu)\Gamma(1/2+\nu)}\, f_{\iota(\nu)\alpha}(z).$$

This is (13) in Proposition 2.8. (Use (33).)

Modular case. Let $\Gamma = \Gamma_{\mathrm{mod}}$ and $\nu \in \mathbb{C} \setminus \left(\frac{1}{2} + \mathbb{Z}\right)$. We have the following equalities:

$$\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi, 0)(z) = \frac{-i}{\sqrt{\pi}}\, \frac{\Gamma(1/2-\nu)}{\Gamma(1-\nu)}\, f_{\iota(\nu)\alpha}|_{1-2\nu}(1-\tilde{k}(-\pi)) \quad \text{(just proved)},$$

$$\mathsf{P}c_{\alpha,\mathrm{gd}}(-\pi, 0) = \mathsf{P}c_{\alpha,\mathrm{gd}}\left(-\frac{\pi}{2}, 0\right)\Big|_{1-2\nu}\left(1+\tilde{k}\left(-\frac{\pi}{2}\right)\right) \quad \text{(cocycle relation)},$$

$$\psi_{\iota(\nu)\alpha} = f_{\iota(\nu)\alpha}|_{1-2\nu}\left(1 - \tilde{k}\left(-\frac{\pi}{2}\right)\right) \qquad \text{(Proposition 2.3).}$$

As $\nu \notin \frac{1}{2} + \mathbb{Z}$, the map $\mathscr{H}_{1-2\nu} \to \mathscr{H}_{1-2\nu} : h \mapsto h_{1-2\nu}(1 - \tilde{k}(-\pi))$ is invertible. Write it as the product of the commuting operators

$$h \mapsto h|_{1-2\nu}(1 + \tilde{k}(-\pi/2)), \quad h \mapsto h|_{1-2\nu}(1 - \tilde{k}(-\pi/2)),$$

to prove the relation (14) in Proposition 2.8. □

References

[1] *R.W. Bruggeman*, Families of Automorphic Forms, Monogr. Math. **88**, Birkhäuser, 1994.
[2] *U. Bunke* and *M. Olbrich*, Gamma-cohomology and the Selberg zeta function, J. reine angew. Math. **467** (1995), 199–219.
[3] *S. Helgason*, Harmonic analysis in the non-Euclidean disk, Conference on Harmonic Analysis, College Park, Maryland, 1971, Springer Lect. Notes Math. **266** (1972), 151–156.
[4] *S. Lang*, $SL_2(\mathbb{R})$, Addison-Wesley, 1975.
[5] *J.B. Lewis*, Spaces of Holomorphic Functions equivalent to the even Maass Cusp Forms, Invent. math. **127** (1997), 271–306.
[6] *J. Lewis* and *D. Zagier*, Period functions and the Selberg zeta function for the modular group, Preprint MPI 96-112, Bonn 1996.
[7] *J.B. Lewis* and *D. Zagier*, Period functions for Maass wave forms, in preparation.
[8] *H. Schlichtkrull*, Hyperfunctions and Harmonic Analysis on Symmetric Spaces, Birkhäuser, 1984.
[9] *G. Shimura*, Introduction to the Arithmetic Theory of Automorphic Functions, Iwanami Shoten, Publishers, and Princeton University Press, 1971.
[10] *L.J. Slater*, Confluent Hypergeometric Functions, University press, Cambridge 1960.
[11] *D. Zagier*, Periods of Modular Forms, Traces of Hecke Operators, and Multiple Zeta Values, Proceedings of the conference Studies of Automorphic forms and *L*-functions, R.I.M.S., Kyoto (1992), 162–170.

Mathematisch Instituut Universiteit Utrecht, Postbus 80010, NL-3508 TA Utrecht
e-mail: bruggeman@math.ruu.nl

Eingegangen 10. Januar 1997