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SUM FORMULA FOR $\mathbf{SL_2}$ OVER A NUMBER FIELD AND SELBERG TYPE ESTIMATE FOR EXCEPTIONAL EIGENVALUES

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1 Introduction

Let F be a number field, and let $\mathcal{O}$ be its ring of integers. We consider the algebraic group $\mathbf{G} = R_{F/\mathbb{Q}}(\mathrm{SL}_2)$ over $\mathbb{Q}$ obtained by restriction of scalars applied to SL_2 over F.

Let $\sigma_1, \ldots, \sigma_d$ be the embeddings $F \to \mathbb{R}$, and $\sigma_{d+1}, \overline{\sigma_{d+1}}, \ldots, \sigma_{d+e}, \overline{\sigma_{d+e}}$ the complex embeddings. We have

$$\begin{aligned} G := \mathbf{G}_{\mathbb{R}} &\cong \mathrm{SL}_2(\mathbb{R})^d \times \mathrm{SL}_2(\mathbb{C})^e \\ \mathbf{G}_{\mathbb{Q}} &\cong \left\{ (x^{\sigma_1}, \ldots, x^{\sigma_d}, x^{\sigma_{d+1}}, \ldots, x^{\sigma_{d+e}}) : x \in \mathrm{SL}_2(F) \right\} . \end{aligned} \tag{1}$$

The image of $\mathrm{SL}_2(\mathcal{O}) \subset \mathrm{SL}_2(F)$ corresponds to $\mathbf{G}_{\mathbb{Z}}$. This is a discrete subgroup of $\mathbf{G}_{\mathbb{R}}$ with finite covolume. We consider the subgroup of finite index $\Gamma = \Gamma_I$ corresponding to $\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) : c \in I \right\}$, where $I \subset \mathcal{O}$ is a non-zero ideal.

Let K be the maximal compact subgroup $\mathrm{SO}_2(\mathbb{R})^d \times \mathrm{SU}_2(\mathbb{C})^e$ of $G = \mathbf{G}_{\mathbb{R}}$. The Hilbert space $L^2(\Gamma\backslash G/K)$ contains $L^2_d(\Gamma\backslash G/K)$ which is the closure of the span of the K-finite vectors of the irreducible G-invariant subspaces of $L^2(\Gamma\backslash G)$. The orthogonal complement of $L^2_d(\Gamma\backslash G/K)$ in $L^2(\Gamma\backslash G/K)$ is described by integrals (or "wave-packets") of Eisenstein series. Let us choose a complete orthonormal basis, $\{\psi_\ell\}_{\ell \geq 0}$ of $L^2_d(\Gamma\backslash G/K)$ such that each ψ_ℓ generates an irreducible subspace of $L^2(\Gamma\backslash G)$. In particular each we call ψ_0.

The decomposition $G/K = (\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}))^d \times (\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C}))^e$ gives rise to Laplace operators $L_1, \ldots, L_{d+e}$ in each of the factors. Each ψ_ℓ is a simultaneous eigenvector of the L_j, with *eigenvalue* $\lambda_\ell = (\lambda_{\ell,1}, \ldots, \lambda_{\ell,d+e}) \in [0,\infty)^{d+e}$. So $\lambda_0 = (0, \ldots, 0)$.

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We normalize the Laplace operators in such a way that the j-th component of the representation generated by ψ_ℓ is in the unitary principal series if and only if

$$\lambda_{\ell,j} \geq \begin{cases} \frac{1}{4} & \text{for } 1 \leq j \leq d\,, \\ 1 & \text{for } d+1 \leq j \leq d+e\,. \end{cases}$$

So the complementary series corresponds to $[0,1/4)$ at the real places, and to $[0,1)$ at the complex places.

The sum formula of Kuznetsov, see [K, Theorems 1 and 2], gives a relation between a bilinear form in Fourier coefficients of real analytic modular forms and a sum of Kloosterman sums. It has been generalized in several ways. Miatello and Wallach, [MW2], have given a formula of the same type for Lie groups of real rank one. The case of a product of rank one groups, of which our group G is an example, is treated in [WM] and [M] (see Theorem 2.1), but the class of test functions in it is too restricted for many applications.

One of the main goals of the present paper is to prove an extension of the sum formula (Theorem 6.1) for a rather general family of test functions. The main new ingredients in the proof are a lemma on the summation over the units of $\mathcal{O}_F$ (Lemma 8.1) and a study of the various Bessel-like transforms occurring in the formula (see sections 5 and 11).

As the main application, we shall give a proof of a Selberg type estimate for exceptional eigenvalues in the context above. Namely, we will prove that for $\ell \geq 1$

$$\lambda_{\ell,j} \geq \begin{cases} 3/16 & \text{if } 1 \leq j \leq d\,, \\ 3/4 & \text{if } d+1 \leq j \leq d+e\,. \end{cases} \tag{2}$$

In the case when $F = \mathbb{Q}$ and Γ is any congruence subgroup of $\mathrm{SL}_2(\mathbb{R})$, the above bound was proved by Selberg in [S], by introducing the (nowadays called) Kloosterman-Selberg zeta function. In the case of any number field F, Gelbart-Jacquet proved a lifting result from forms on GL(2) to forms on GL(3) that implies the strict inequality in (2) (see [GJ, Theorem 9.3 (4)]). They use machinery on L-functions, in particular many results from [GoJ], [JPSa]. In a preprint, [LuRS2], Luo, Rudnick and Sarnak obtain even stronger results by L-function methods. Our method here is very different and more elementary. Besides the spectral decomposition, we use the Salié-Weil type estimate of Kloosterman sums over a number field given in [BM, Theorem 10], and analysis of certain classical special functions. In the case $F = \mathbb{Q}$ our result gives another proof of Selberg's lower bound, by using as a main tool, the Kuznetsov sum formula in place of the Kloosterman-Selberg zeta function. This zeta function is not available in the context of this paper. Of course, [GJ] contains more information than estimate (2) above, (in particular we only get information at the infinite places), but we feel our alternative proof gives an interesting application of the sum formula.

Selberg has conjectured that actually $\lambda_1 \geq 1/4$ (a bound that had been established by Roelcke for the modular group). Progress towards Selberg's conjecture has been made by various authors, notably by Iwaniec ([I]), for Hecke congruence groups, Gelbart-Jacquet ([GJ]), and by Luo-Rudnick-Sarnak ([LuRS1]), who proved the best known bound, $\lambda_1 \geq 21/100$, valid for any congruence subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Overview. In the sections 2–5, we prepare the statement of the sum formula, Theorem 6.1. Section 7 contains the proof of the inequalities (2), and gives an estimate of the measure occurring in the spectral side of the sum formula. We prove Theorem 6.1 in the sections 8–10. In section 11 we discuss the estimates of Bessel functions that we use throughout this paper.

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2 Notations and Conventions

We consider F as embedded in $\mathbb{R}^d \times \mathbb{C}^e$ by $\xi \mapsto (\xi^{\sigma_1}, \dots, \xi^{\sigma_{d+e}})$, and similarly, $\mathbf{G}_{\mathbb{Q}} \subset G = \mathrm{SL}_2(\mathbb{R})^d \times \mathrm{SL}_2(\mathbb{C})^e$.

We take $n_j := 1$ for $1 \leq j \leq d$, and $n_j = 2$ for $d+1 \leq j \leq d+e$. For $x \in \mathbb{R}^d \times \mathbb{C}^e$, we put $S(x) := \sum_{j=1}^{d+e} n_j \operatorname{Re} x_j$. This extends the trace $\mathrm{Tr}_{F/\mathbb{Q}} : F \to \mathbb{Q}$. Similarly, $N(y) := \prod_{j=1}^{d} y_j \prod_{j=d+1}^{d+e} |y_j|^2$ extends the norm of F over $\mathbb{Q}$ to $(\mathbb{R}^*)^d \times (\mathbb{C}^*)^e$.

If $x, u \in \mathbb{R}^d \times \mathbb{C}^e$, we define $xu \in \mathbb{R}^d \times \mathbb{C}^e$ by $(xu)_j := x_j u_j$. The $\mathbb{R}$-linear forms on $\mathbb{R}^d \times \mathbb{C}^e$ are $x \mapsto S(qx)$, with q running through $\mathbb{R}^d \times \mathbb{C}^e$.

Functions of product type. The test functions on G that we use are mostly of product type: $f(g) = \prod_{j=1}^{d+e} f_j(g_j)$ for $g = (g_1, \dots, g_d) \in G$, with f_j a complex valued function on the j-th factor $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$. We use the notation $f = \times_{j=1}^{d} f_j$. We use the same concept of product type, and the $\times$-notation, for functions on $\mathbb{C}^{d+e}$ and other products.

Subgroups of G. For $y \in \mathbb{R}_{>0}^{d+e}$ we put $a[y] := (a_1[y_1], ..., a_{d+e}[y_{d+e}]) \in G$, with $a_j[y_j] = \begin{pmatrix} \sqrt{y_j} & 0 \\ 0 & 1/\sqrt{y_j} \end{pmatrix}$. This defines an injective group homomorphism from the multiplicative group $\mathbb{R}_{>0}^{d+e}$ into G. The image A is the identity component of a maximal $\mathbb{R}$-split torus in G. We normalize the Haar measure on A by $da = \frac{dy_1}{y_1} \cdots \frac{dy_{d+e}}{y_{d+e}}$ for $a = a[y]$. We define ${}^0A := \{ a[y] : y \in \mathbb{R}_{>0}^{d+e},\ N(y) = 1 \}$.

For $x \in \mathbb{R}^d \times \mathbb{C}^e$ we define $n[x] := \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & x_{d+e} \\ 0 & 1 \end{pmatrix} \right) \in G$. The group $N := \{ n[x] : x \in \mathbb{R}^d \times \mathbb{C}^e \}$ is unipotent. The normalization $dn = \frac{dx_1}{\pi} \cdots \frac{dx_d}{\pi} \frac{dx_{d+1}\, d\overline{x_{d+1}}}{-2\pi i} \cdots \frac{dx_{d+e}\, d\overline{x_{d+e}}}{-2\pi i}$ for $n = n[x]$ of the Haar measure on N is the one prescribed in [WM, p. 309].

For $u \in (\mathbb{R}^*)^d \times (\mathbb{C}^*)^e$ we define $b[u] := \left(\begin{pmatrix} u_1 & 0 \\ 0 & 1/u_1 \end{pmatrix}, \dots, \begin{pmatrix} u_{d+e} & 0 \\ 0 & 1/u_{d+e} \end{pmatrix} \right)$. So $a[y] = b[\sqrt{y}]$ for $y \in \mathbb{R}_{>0}^{d+e}$, with $(\sqrt{y})_j = \sqrt{y_j}$.

Let M be the subgroup $\{ b[u] : |u_j| = 1 \text{ for all } j \}$ of K. We call $P := NAM$ the *standard parabolic subgroup* of G. The Iwasawa decomposition $G = NAK$ leads to $g = n(g)a(g)k(g)$.

Roots. The simple roots α_j satisfy $a[y]^{\alpha_j} = y_j$. We identify $\mathbb{C}^{d+e}$ with the complexified Lie algebra $\mathfrak{a}_{\mathbb{C}}^*$ of A by $a[y]^\nu = \prod_{j=1}^{d+e} y_j^{\nu_j}$. We use $\rho := \sum_{j=1}^{d+e} \frac{1}{2} n_j \alpha_j$. Note that $a[y]^\rho = N(y)^{1/2}$.

We choose $a^{-2\rho}\, dn\, da$ as the left invariant measure on G/K.

Discrete subgroups. Put $\Gamma_N := \Gamma \cap N$ and $\Gamma_P := \Gamma \cap P$. For our special choice of Γ, we can take the elements $b[\varepsilon]$ with $\varepsilon \in \mathcal{O}^*$ as representatives of $\Gamma_N \backslash \Gamma_P$. The group $\{ b[\varepsilon] : \varepsilon \in \mathcal{O}^* \}$ is contained in ${}^0AM \subset AM$, not in A.

Characters. We put $\mathcal{O}' := \{ y \in F : \mathrm{Tr}(yx) \in \mathbb{Z} \text{ for all } x \in \mathcal{O} \}$. So $\mathcal{O}'$ is a fractional ideal containing the ring of integers $\mathcal{O}$.

The unitary characters of N are of the form $n[x] \mapsto e^{2\pi i S(qx)}$ with $q \in \mathbb{R}^d \times \mathbb{C}^e$. Such a character is trivial on Γ_N if and only if $q = (r^{\sigma_1}, \dots, r^{\sigma_{d+e}})$ with $r \in \mathcal{O}'$. We denote such a character by χ_r. Note that $\chi_r(n[\xi]) = e^{2\pi i\, \mathrm{Tr}(r\xi)}$ if $\xi \in F$. Furthermore, if $r \in \mathcal{O}' \smallsetminus \{0\}$, then the character χ_r is non-trivial at all places $j = 1, \dots, d+e$.

For each $r \in \mathcal{O}' \smallsetminus \{0\}$ we define $a_r \in A$ by $a_r = a[\xi_r]$, where $\xi_{r,j} := 2\pi n_j |r^{\sigma_j}|$. This element a_r is denoted by a_χ in [WM], if $\chi = \chi_r$.

***N*-equivariant eigenfunctions.** Let $r \in \mathcal{O}'$, $r \neq 0$. If a function f on G/K satisfies $f(ngk) = \chi_r(n) f(g)$ for all $n \in N$ and $k \in K$, and is also an eigenfunction of each Laplace operator L_j with eigenvalue $\frac{1}{4}n_j^2 - \nu_j^2$, then it is an element of a linear space of dimension $2\,(d+e)$. If we impose moreover that $f(a[y])$ be bounded for $y_j \geq 1$ for all j, then f is a multiple of the function $\mathsf{W}_{\nu,r}$ defined by

$$\mathsf{W}_{\nu,r}(na[y]k) := \chi_r(n) a[y]^\rho \prod_{j=1}^{d+e} K_{\nu_j}(y_j \xi_{r,j})\,. \tag{3}$$

The modified Bessel function K_ν is given in (36).

In (5) and (3) of [WM], the function $J^{\chi_r}(\nu, g) = d_r(\nu)\mathsf{W}_{r,\nu}(g)$ is used, with

$$d_r(\nu) := \prod_{j=1}^{d+e} d_{r,j}(\nu_j) \tag{4}$$

$$d_{r,j}(\nu) := \begin{cases} \pi^{-1/2} 2^{1-\nu} \xi_{r,j}^\nu \, \Gamma\left(\nu + 1/2\right)^{-1} & \text{if } j \leq d\,, \\ 2^{1-\nu} \xi_{r,j}^\nu \, \Gamma\left(\nu+1\right)^{-1} & \text{if } j > d\,. \end{cases}$$

3 Automorphic Forms

Spectral parameters. Let $\ell \geq 0$. We define $\mu_\ell \in \prod_{j=1}^{d+e} \left(i[0,\infty) \cup \left(0, \frac{1}{2}n_j\right]\right)$ by $\lambda_{\ell,j} = \left(\frac{1}{2}n_j\right)^2 - \mu_{\ell,j}^2$.

Each ψ_ℓ in the orthonormal basis of $L^2_d(\Gamma\backslash G/K)$ is determined by its values on NA. The Γ-invariance implies that ψ_ℓ is given by its Fourier expansion

$$\begin{aligned} \psi_\ell(nak) =& F_0\psi_\ell(a) \\ &+ \operatorname{vol}(\Gamma_N\backslash N)^{-1} \sum_{r \in \mathcal{O}',\, r\neq 0} c_r(\psi_\ell) d_r(\overline{\mu_\ell}) \chi_r\left(n\right) \mathsf{W}_{\mu_\ell, r}(a)\,, \end{aligned} \tag{5}$$

for $n \in N$, $a \in A$, $k \in K$; see (3) and (4). We do not make explicit the Fourier term $F_0\psi_\ell$ of order zero. The factor $d_r(\nu)$ is non-zero for the relevant values of ν. The following fact is standard:

LEMMA 3.1. *If $\psi_\ell(na) = F_0\psi_l(a)$, then ψ_ℓ is constant.*

Proof. Suppose $\psi_\ell(n[x]a[y]) = F_0\psi_l(a[y])$. Take some $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma$ with $\gamma \neq 0$. Take $x = (0, \ldots, 0, x_{\tilde{\jmath}}, 0, \ldots, 0)$ and let $y \in \mathbb{R}^{d+e}_{>0}$. The Γ-invariance implies $F_0\psi_\ell(a[y]) = F_0\psi_\ell(a[\tilde{y}])$ with $\tilde{y}_j = y_j/(|y_j\gamma^{\sigma_j}|^2 + |\delta^{\sigma_j}|^2)$ for $j \neq \tilde{\jmath}$

and $\tilde y_{\tilde\jmath} = y_{\tilde\jmath}/(|y_{\tilde\jmath}\gamma^{\sigma_{\tilde\jmath}}|^2 + |x_{\tilde\jmath}\gamma^{\sigma_{\tilde\jmath}} + \delta^{\sigma_{\tilde\jmath}}|^2)$. The dependence on x_j shows that $F_0\psi_\ell(a[y])$ does not depend on $y_{\tilde\jmath}$. This holds for $\tilde\jmath = 1,\dots,d+e$. □

Definition 3.2. Define $Y := \prod_{j=1}^{d+e}\left(i[0,\infty)\cup\left(0,\frac12 n_j\right)\right)$.

Lemma 3.3. *If $\ell \neq 0$, then $\mu_\ell \in Y$.*

Proof. Suppose that $\mu_{\ell,j} = \frac12 n_j$. This implies that the representation generated by ψ_ℓ is trivial at the place j. Hence $\psi_\ell(g)$ does not depend on the j-th factor g_j of $g \in G$. So for each $r \in \mathcal{O}'$, $r \neq 0$, we have

$$\int_{N/\Gamma_N} \chi_r\big(n(x)\big)^{-1}\psi_\ell\big(n(x)a(y)\big)dx = 0\,.$$

(Integrate first over the j-th factor. As $r^{\sigma_j} \neq 0$, this yields 0.) So $c_r(\psi_\ell) = 0$ for all $r \in \mathcal{O}' \smallsetminus \{0\}$. Application of Lemma 3.1 completes the proof. □

Eisenstein series. The $\mathbb{Q}$-rank of $\mathbf{G}$ is one. All Γ-percuspidal parabolic subgroups P' of G are of the form $P' = qPq^{-1}$, with $q \in \mathbf{G}_\mathbb{Q}$ (take $q = 1$ for $P' = P$). We write $q = kp$, $p \in P$, $k \in K$; so $P' = kPk^{-1}$. There is a decomposition $P' = N'A_0'L'$, with $N' = kNk^{-1}$, $A_0' = \big\{ka[y]k^{-1} : y_1 = y_2 = \cdots = y_{d+e} > 0\big\}$, $L' = {}^0A'\,M$, with ${}^0A' = \big\{ka[y]k^{-1} : N(y) = 1\big\}$. We describe the characters of $A_0'\,{}^0A'$ by $a_\mathbb{C}^* \cong \mathbb{C}^{d+e}$ via $(ka[y]k^{-1})^\mu = \prod_{j=1}^{d+e} y_j^{\mu_j}$. In this way all characters of A_0' can be identified to a complex multiple of ρ, and all characters of ${}^0A'$ are given by $\mu \in \mathbb{C}^{d+e}$ with $S(\mu) = 0$. For each $g \in G$, there is a unique $a'(g) \in A_0'\,{}^0A'$ such that $g \in N'a'(g)K$. In the terminology of [H, Chap. I, §1], the (P', A_0') are mincuspidal pairs.

The Eisenstein series

$$E(P',\nu,i\mu,g) := \sum_{\gamma\in\Gamma_{P'}\backslash\Gamma} a'(\gamma g)^{\rho+\nu+i\mu}$$

depends on $\nu \in \mathbb{C}$, and on $\mu \in \mathbb{R}^{d+e}$ chosen in such a way that $a^\mu = 1$ for $a \in A_0'$ and $a'(\gamma)^\mu = 1$ for all $\gamma \in \Gamma_{P'} := \Gamma \cap P'$. So the μ are in a lattice in the hyperplane $S(x) = 0$ in $\mathbb{R}^{d+e}$.

The Eisenstein series converges absolutely for $\nu = z\rho$, $\operatorname{Re} z > 1$. There is a meromorphic continuation in $\nu \in \mathbb{C}\rho$, giving $\nu \mapsto E(P',\nu,i\mu)$ as a family of automorphic forms. (For the convergence, see, e.g., [H, p. 29, and Lemma 23, and for the continuation, Theorem 7 on p. 105.])

There is a Fourier expansion for each $E(P',\nu,i\mu,g)$ of the same type as for ψ_ℓ in (5). The Fourier coefficients $c_r(\psi_\ell)$ are replaced by $D_r\,(P',\nu,i\mu)$. Let $P^1,\dots,P^m$ be a system of representatives of Γ-conjugacy classes of percuspidal parabolic subgroups. The $D_r(P^j,\nu,i\mu)$ correspond to the functions described in [WM, Lemma 5.2] on the region of absolute convergence.

The discussion in Chap. I, §3 of [H] shows that we know the growth behavior of an automorphic form if we know it on Siegel domains for all P^i. A Siegel domain $\mathfrak{S}$ for P' is a set of the form $\mathfrak{S}=\omega A_0'(t)K$, with a bounded set $\omega\subset N'L'$ and $A_0'(t)=\left\{a\in A_0' : a^\rho\ge t\right\}$.

Eisenstein series have polynomial growth (Lemma 24 of [H, p. 30]). The Fourier expansion for P' shows that the growth behavior on a Siegel domain $\mathfrak{S}$ for P' is determined by the Fourier term of order zero. Theorem 5 in [H, p. 44] gives the form of these first order Fourier terms. We conclude that $E(P^j,z\rho,i\mu)$, for $\operatorname{Re} z>1$, is bounded on a Siegel domain $\mathfrak{S}$ for P' if P^j is not of the form $\gamma P'\gamma^{-1}$ with $\gamma\in\Gamma$, and $E(P^j,z\rho,i\mu,g)-a'(g)^{\rho+z\rho+i\mu}$ is bounded for $g\in\mathfrak{S}$ if P^j and P' are conjugate under Γ.

Spectral decomposition. For continuous bounded functions f, f_1 on $\Gamma\backslash G/K$ we have the following description of the scalar product:

$$\begin{aligned}\langle f,f_1\rangle &= \sum_{\ell\ge 0}\langle f,\psi_\ell\rangle\,\overline{\langle f_1,\psi_\ell\rangle} \\ &\quad+\sum_{j=1}^{m}\sum_{\mu} c_j\int_{-\infty}^{\infty}\big\langle f,E(P^j,iy\rho,i\mu)\big\rangle\,\overline{\big\langle f_1,E(P^j,iy\rho,i\mu)\big\rangle}\,dy\,,\end{aligned}\tag{6}$$

with suitable positive constants c_j. The μ run through a lattice depending on P^j. Note that $E(P^j,iy\rho,i\mu)$ is integrable. Hence the integral defining $\big\langle f,E(P^j,iy\rho,i\mu)\big\rangle$ converges absolutely.

DEFINITION 3.4. For $r,r'\in\mathcal{O}'\smallsetminus\{0\}$ we define a measure $d\sigma_{r,r'}$ on Y by

$$\begin{aligned}\int_Y\eta(\nu)\,d\sigma_{r,r'}(\nu) &:= \sum_{\ell\ge 1}\overline{c_r(\psi_\ell)}c_{r'}(\psi_\ell)d_r(\mu_\ell)\overline{d_{r'}(\mu_\ell)}\eta(\mu_\ell) \\ &\quad+\sum_{j=1}^{m}c_j\sum_{\mu}\int_{-\infty}^{\infty}\overline{D_r(P^j,iy\rho,i\mu)}D_{r'}(P^j,iy\rho,i\mu) \\ &\qquad\cdot d_r(iy\rho+i\mu)\overline{d_{r'}(iy\rho+i\mu)}\eta(iy\rho+i\mu)\,dy.\end{aligned}$$

We apply this only to even functions η, so we can view $d\sigma_{r,r'}$ as a measure on Y.

This measure is one of the ingredients of the sum formula. It encodes information on Fourier coefficients of automorphic forms.

4 Kloosterman Sums

The other main ingredient of the sum formula is a sum involving Kloosterman sums for the number field F.

Definition 4.1. For $c \in \mathcal{O} \smallsetminus \{0\}$ and $r, r' \in \mathcal{O}' \smallsetminus \{0\}$ we define the *Kloosterman sum* by

$$S(r, r'; c) := \sum_{d \in (\mathcal{O}/(c))^*} e^{2\pi i \operatorname{Tr}((rd + r'a)/c)}.$$

Here $a \in \mathcal{O}$ is such that $ad \equiv 1 \bmod (c)$, and $\operatorname{Tr} = \operatorname{Tr}_{F/\mathbb{Q}}$.

For $F = \mathbb{Q}$ one obtains the classical Kloosterman sums. A trivial estimate of the Kloosterman sum is $|S(r, r'; c)| \le |N(c)| = |\mathcal{O}/(c)|$. In Theorem 10 of [BM], we have shown that the classical Weil-Salié estimate can be generalized to this situation:

$$\left|S(r, r'; c)\right| \le 2^{d+e+f/2} \sqrt{|D_F|} \sqrt{N_{r,r'}(c)} 2^{\operatorname{pr}(c)} \sqrt{|N(c)|}\,,$$

with f the sum of the residue class degrees of the primes of F above (2), D_F the discriminant of F, $\operatorname{pr}(c)$ the number of prime ideals dividing the ideal (c), $N_{r,r'}(c)$ a number (bounded as a function of c) that depends on r, r'. (In the case $F = \mathbb{Q}$, this is $S(r, r_1; c) \le 2\sqrt{2}\, 2^{\operatorname{pr}(c)} c^{1/2} (r, r_1, c)^{1/2}$. In [BM, p. 130] we omitted the factor $2\sqrt{2}$.) This implies (see [BM, remark after Theorem 10])

$$S(r, r'; c) \ll_{F, r, r', \delta} |N(c)|^{1/2+\delta} \tag{7}$$

for each $\delta > 0$.

Definition 4.2. Let $\Gamma = \Gamma_I$, $I \subset \mathcal{O}$ a non-zero ideal, and $r, r' \in \mathcal{O}' \smallsetminus \{0\}$. If $f : (\mathbb{R}^*)^d \times (\mathbb{C}^*)^e \to \mathbb{C}$, we define the *Kloosterman term*

$$K_{r,r'}(f) := \sum_{c \in I,\, c \ne 0} \frac{S(r, r'; c)}{|N(c)|} f\left(\frac{rr'}{c^2}\right).$$

The Kloosterman term converges absolutely if f has compact support in $(\mathbb{R}^*)^d \times (\mathbb{C}^*)^e$. Using (7), one can figure out a larger class of functions for which $K_{r,r'}(f)$ converges.

5 Test Functions and Delta Term

The test functions in the sum formula are holomorphic functions on a subset of $\mathbb{C}^{d+e}$. They will be integrated with respect to the measure $d\sigma_{r,r'}$. The functions to be inserted in the Kloosterman term are obtained by a "Bessel transformation".

DEFINITION 5.1. For $\sigma > 0$ and $a = (a_1, \ldots, a_{d+e})$, with $a_j > 2$ for $j = 1, \ldots, d+e$, we define $\mathcal{K}_a(\sigma)$ as the set of functions $k = \times_{j=1}^{d+e}$, where each k_j is an even holomorphic function on $|\operatorname{Re}\nu| \leq n_j\sigma$ that satisfies

$$k_j(\nu) \ll e^{-\pi|\operatorname{Im}\nu|}\bigl(1 + |\operatorname{Im}\nu|\bigr)^{-a_j}.$$

DEFINITION 5.2. For functions f that are even in each coordinate, we define

$$\int_Y f(\nu)\, d\eta(\nu) := \prod_{j=1}^{d+e} \frac{i}{2} \int_{\operatorname{Re}\nu_j = 0} f_j(\nu_j)\, \nu_j \sin \pi\nu_j\, d\nu_j\,. \tag{8}$$

This defines a positive measure $d\eta$ on the set $(i[0,\infty))^{d+e} \subset Y = \prod_{j=1}^{d+e} (i[0,\infty) \cup (0, n_j/2))$.

Each $k \in \mathcal{K}_a(\sigma)$ is integrable for $d\eta$.

DEFINITION 5.3. For $r, r' \in \mathcal{O}' \setminus \{0\}$ we put

$$\alpha(r, r') := (2/\pi^2)^{d+e} \operatorname{vol}(\Gamma_N \backslash N) \cdot \bigl| \bigl\{\varepsilon \in \mathcal{O}^* : r = \varepsilon^2 r'\bigr\} \bigr|\,.$$

For most pairs (r, r') we have $\alpha(r, r') = 0$. If $\alpha(r, r') \neq 0$, then $\operatorname{sign}(r_j) = \operatorname{sign}(r'_j)$ for all $j = 1, \ldots, d$, and there are exactly two $\varepsilon \in \mathcal{O}^*$ such that $r = \varepsilon^2 r'$. (We have defined $(\varepsilon^2 r')_j = (\varepsilon^{\sigma_j})^2 r'_j$.)

As $n(\gamma) \in \Gamma_N$ for all $\gamma \in \Gamma_P$ for our group Γ, this definition is simpler than the corresponding one in Theorem 6.1 of [WM].

DEFINITION 5.4. For $r, r' \in \mathcal{O}' \setminus \{0\}$, and $k \in \mathcal{K}_a(\sigma)$, we define the *delta term* by

$$\Delta_{r,r'}(k) := \alpha(r, r') \int_Y k(\nu)\, d\eta(\nu)\,.$$

This converges absolutely.

DEFINITION 5.5. $\mathcal{B} := \times_{j=1}^{d+e} \mathcal{B}_j$ is the function on $(\mathbb{R}^* \times \mathbb{C})^d \times (\mathbb{C}^* \times \mathbb{C})^e$ given by

$$\mathcal{B}_j(t, \nu) := \frac{1}{\sin \pi\nu} \bigl(\mathcal{J}_j(t, -\nu) - \mathcal{J}_j(t, \nu)\bigr)\,,$$
$$\mathcal{J}_j(t, \nu) := \begin{cases} J_{2\nu}^{\operatorname{sign}(t)}\bigl(4\pi\sqrt{|t|}\,\bigr) & \text{if } j \leq d\,, \\ I_\nu\bigl(4\pi\sqrt{t}\bigr) I_\nu\bigl(4\pi\overline{\sqrt{t}}\,\bigr) & \text{if } j > d\,, \end{cases}$$

with $t \in (\mathbb{R}^*)^d \times (\mathbb{C}^*)^e$ and $\nu \in \mathbb{C}^{d+e}$.

See (34)–(37) for the Bessel functions I_ν and $J_{2\nu}^{\pm 1}$. In the case $j > d$, the choice of the square root of t does not matter, provided we take $\arg \overline{\sqrt{t}} = -\arg \sqrt{t}$.

LEMMA 5.6. *The function $\nu \mapsto \mathcal{B}(t,\nu)$ is even in each variable and holomorphic on $\mathbb{C}^{d+e}$ for each $t \in (\mathbb{R}^*)^d \times (\mathbb{C}^*)^e$, and satisfies the relation $\overline{\mathcal{B}(t,\bar{\nu})} = \mathcal{B}(t,\nu)$. The functions $\mathcal{J}$ and $\mathcal{B}$ satisfy the estimates* (9)–(15), *with $\alpha = \operatorname{Re}\nu$ fixed, and $t_0 > 0$.*

$$\begin{aligned} &\mathcal{J}_j(t,\nu) \\ &\quad \ll_{\alpha,t_0} e^{\pi|\operatorname{Im}\nu|}\bigl(1+|\operatorname{Im}\nu|\bigr)^{-2\alpha-n_j/2}|t|^{\alpha} \text{ for } |t| \le t_0\,, && (9)\\ &\quad \ll_{\alpha} e^{\pi|\operatorname{Im}\nu|} \qquad t \ge 1\,,\ j \le d\,,\ \alpha > 0\,, && (10)\end{aligned}$$

$$\begin{aligned} &\mathcal{B}_j(t,\nu) \\ &\ll_{t_0} \bigl|\log|t|\bigr|\,|t|^{-|\alpha|} && |t| \le t_0\,,\ |\nu| \le 1\,, && (11)\\ &\ll_{\alpha,t_0} \bigl(1+|\operatorname{Im}\nu|\bigr)^{2|\alpha|-n_j/2}|t|^{-|\alpha|} && |t| \le t_0\,,\ \begin{cases} l|\nu| \ge 1 & \text{if } \alpha = 0\,,\\ \text{all } \nu & \text{if } \alpha \ne 0\,,\end{cases} && (12)\\ &\ll_{\alpha} \bigl(1+|\operatorname{Im}\nu|\bigr)^{2\alpha}|t|^{-\alpha} && t \le -1\,,\ j \le d\,,\ \alpha > 0\,, && (13)\\ &\ll_{\alpha} \bigl(1+|\operatorname{Im}\nu|\bigr)^{2\alpha+1}|t|^{-\alpha-1/2} && t \le -1\,,\ j \le d\,,\ \alpha > -\tfrac12\,, && (14)\\ &\ll e^{\frac12\pi|\operatorname{Im}\nu|}\bigl(1+|\operatorname{Im}\nu|\bigr)\,|t|^{-1/4} && |t| \ge 1\,,\ j > d\,,\ \alpha = 0\,,\ |\nu| \ge 1\,. && (15)\end{aligned}$$

Proof. Most assertions are clear. For the estimates, we use Lemma 11.1 in the following way:

estimate:	(9)	(10)	(11)	(12)	(13)	(14)	(15)
use:	(38), (39)	(40)	(34), (35), (37)	(9)	(41)	(42)	(43)

$\square$

DEFINITION 5.7. For each $k \in \mathcal{K}_a(\sigma)$ we define the *Bessel transform Bk* on $(\mathbb{R}^*)^d \times (\mathbb{C}^*)^e$ by

$$Bk(t) := \int_Y k(\nu)\mathcal{B}(t,\nu)\,d\eta(\nu)\,.$$

LEMMA 5.8. *The integral defining $Bk(t)$ converges absolutely. Furthermore, we have $Bk = \times_{j=1}^{d+e}\beta_j k_j$, with*

$$\begin{aligned}\beta_j k_j(t) &= \frac{i}{2}\int_{\operatorname{Re}\nu=\alpha_1} k_j(\nu)\mathcal{B}_j(t,\nu)\,\nu\sin\pi\nu\,d\nu && (16)\\ &\qquad \text{with } |\alpha_1| \le n_j\sigma \text{ and } |\alpha_1| < \tfrac12 a_j + \tfrac14 n_j - 1\\ &= -i\int_{\operatorname{Re}\nu=\alpha_2} k_j(\nu)\mathcal{J}_j(t,\nu)\,\nu\,d\nu && (17)\\ &\qquad \text{with } |\alpha_2| \le n_j\sigma \text{ and } \alpha_2 > 1 - \tfrac14 n_j - \tfrac12 a_j\end{aligned}$$

for $t \in \mathbb{R}^*$ if $j \le d$, and $t \in \mathbb{C}^*$ if $j > d$.

Proof. The convergence of (16) with $\alpha_1 = 0$ follows from (11) and (12). This gives the convergence in Definition 5.7. Use the definition of $\mathcal{B}_j$, estimate (9), and $k_j(-\nu) = k_j(\nu)$ to obtain (17) with $\alpha_2 = 0$.

Estimate (9) shows that (17) holds for any $\alpha_2 > 1 - \frac{1}{4}n_j - \frac{1}{2}a_j$ such that the line of integration is contained in the domain of k_j. Similarly, (12) gives the condition $|\alpha_1| < \frac{1}{2}a_j + \frac{1}{4}n_j - 1$ for (16). □

6 Sum Formula

We are now ready to state the sum formula of Kuznetsov for $\Gamma\backslash G/K$.

Theorem 6.1. *Choose $\sigma \in \left(\frac{1}{2}, 1\right)$, and $a \in \mathbb{R}^{d+e}_{>0}$ such that $a_j > 4 + n_j$ for $j = 1, \ldots, d+e$. Let $r, r' \in \mathcal{O}' \smallsetminus \{0\}$, and $k \in \mathcal{K}_a(\sigma)$. Then k is integrable for the measure $d\sigma_{r,r'}$. The integral in $\Delta_{r,r'}(k)$ and the sum $K_{-r,-r'}(Bk)$ converge absolutely, and*

$$\int_Y k(\nu)\, d\sigma_{r,r'}(\nu) = \Delta_{r,r'}(k) + 2^{d+e}\pi^{-2d-e} K_{-r,-r'}(Bk) \tag{18}$$

$$= \left(\frac{i}{2}\right)^{d+e} \alpha(r,r') \prod_{j=1}^{d+e} \int_{\mathrm{Re}\,\nu_j = 0} k_j(\nu_j)\nu_j \, \sin \pi\nu_j \, d\nu_j + \frac{2^{d+e}}{\pi^{2d+e}} \sum_{c \in I,\, c \neq 0} \frac{S(-r,-r';c)}{|N(c)|} Bk\left(\frac{rr'}{c^2}\right).$$

The measure $d\sigma_{r,r'}$ on $Y = \prod_{j=1}^{d+e}\left(i[0,\infty) \cup (0, n_j/2)\right)$ depends on products of Fourier coefficients of automorphic forms, see Definition 3.4. The *test function* $k \in \mathcal{K}_a(\sigma)$ is holomorphic on the product of the strips $|\mathrm{Re}\,\nu_j| \leq n_j\sigma$, and satisfies a growth condition (Definition 5.1). See Definition 5.7 for the *Bessel transform* Bk of a test function k, and Definition 5.3 for $\alpha(r,r')$.

The theorem gives the absolute convergence of all terms in (18).

For groups of real rank one, the class of test functions can be enlarged to include $a_j > 2$ by an extension step (see [B, p. 185], or the final step in the proof of Theorem 1.9 in [MW2]). For this step, one needs to know the convergence of the Kloosterman term beforehand. In the presence of complex places, we do not know how to obtain that a priori convergence.

We give the *proof* of this theorem in sections 8–10.

Remark. In [Jo], Theorem 4.25, Joyner states the sum formula in the particular case where F is a quadratic totally real number field of class number one, for a subset of $r, r' \in \mathcal{O}$. However, there are points in his two proofs that we do not understand.

For instance, the second proof is based on the Poincaré series considered in (4.4). These are given by a sum over $\Gamma_\infty\backslash\Gamma$. Joyner defines Γ_∞ as our Γ_P. With that interpretation, the series in (4.4) is not well defined. In the interpretation $\Gamma_\infty = \Gamma_N$, the convergence of the series in (4.4) requires a proof.

Also in the first proof, in [Jo, §3], there seems to be a confusion on whether the Kloosterman term is a sum over $c \in \mathcal{O} \smallsetminus \{0\}$, or over the non-zero ideals $(c) \subset \mathcal{O}$. (Note that the Kloosterman sum $S(r, r'; c)$ depends on the choice of $c \in (c)$.)

7 Estimation of the Measure and Proof of the Selberg Type Bound

The Selberg type bounds in (2) are based on the following result:

Proposition 7.1. *Let $r \in \mathcal{O}'$, $r \neq 0$. The support of the measure $d\sigma_{r,r}$ is contained in the subset $\prod_{j=1}^{d+e} (i[0,\infty) \cup (0, n_j/4])$ of Y.*

For $X > 1$, put $Y(X) := \{\nu \in Y : |\nu_j| \leq X \text{ for } j = 1, \ldots, d+e\}$. Then

$$\int_{Y(X)} \left(\prod_{j=1}^{d} \frac{\frac{1}{4} - \nu_j^2}{\cos \pi\nu_j} \cdot \prod_{j=d+1}^{d+e} \frac{\nu_j(1-\nu_j^2)^2}{\sin \pi\nu_j \cos \frac{\pi\nu_j}{2}} \right) d\sigma_{r,r}(\nu) \ll X^{4d+8e} \tag{19}$$

as $X \to \infty$.

Remarks. Let $r, r' \in \mathcal{O}' \smallsetminus \{0\}$. The structure of the measure $d\sigma_{r,r'}$ (see Definition 3.4) implies directly that $\mathrm{supp}(d\sigma_{r,r'})$ is contained in the same subset of $Y = \prod_{j=1}^{d+e} (i([0,\infty) \cup (0, n_j/2))$.

In subsection 7.2 we shall use the sum formula to prove this proposition.

7.1 Proof of a Selberg type estimate. The integral in Definition 3.4, with Fourier coefficients of Eisenstein series, contributes to $d\sigma_{r,r}$ a measure with support in $(i[0,\infty))^{d+e}$. So Proposition 7.1 is a statement concerning the first term in the definition of $d\sigma_{r,r'}$, corresponding to the discrete spectrum. We conclude that, for each $r \in \mathcal{O}' \smallsetminus \{0\}$, the coefficients $c_r(\psi_\ell)$ can be non-zero only if $\mu_{\ell,j} \in i[0,\infty) \cup \left(0, \frac{1}{4}n_j\right]$ for $j = 1, \ldots, d+e$. This is equivalent to $\lambda_{\ell,j} \geq \frac{3}{16}n_j^2$ for $j = 1, \ldots, d+e$. So if ψ_ℓ would violate one of the inequalities in (2), then all its Fourier coefficients $c_r(\psi_\ell)$, with $r \neq 0$, would vanish. Lemma 3.1 shows that such ψ_ℓ have to be constant, hence $\ell = 0$. $\square$

7.2 Proof of Proposition 7.1. We apply Theorem 6.1 with $r = r' \in \mathcal{O}' \setminus \{0\}$. At one point we shall use a convergence result, Lemma 8.1, to be proved in the next section.

It is instructive to follow the arguments in this proof in the case $F = \mathbb{Q}$. This gives an alternative way to obtain the Selberg estimate $\lambda_1 \geq 3/16$.

Test function. We take $k = \times_{j=1}^{d+e} k_j$ depending on $p \in (\mathbb{R}_{>0})^{d+e}$ in the following way:

$$k_j(\nu) := e^{p_j \nu^2} \varphi_j(\nu)\,,$$

with φ_j an even holomorphic function on $|\operatorname{Re}\nu| \leq n_j\sigma$ satisfying $\varphi_j(\nu) \ll e^{-\frac{1}{2}(1+n_j)\pi|\operatorname{Im}\nu|}(1+|\operatorname{Im}\nu|)^{m_j}$ for some $m_j \in \mathbb{N}_{\geq 2}$, and $\varphi_j(\nu) > 0$ for $\nu \in i\mathbb{R} \cup [-n_j\sigma, n_j\sigma]$. The function $\Phi := \times_{j=1}^{d+e} \varphi_j$ is positive on Y. See the integrand in (19) for an example of such a Φ.

The decay of the factor $e^{p_j\nu^2}$ ensures that $k \in \mathcal{K}_a(\sigma)$, with σ and a as in Theorem 6.1. We first estimate the delta term and the Kloosterman term uniformly in p. This estimate depends on a number $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$.

Bessel transform. We consider a factor $\beta_j k_j(y)$ of Bk_j. For $|y| \leq 1$ we use (17) and (9) to obtain for $\alpha_2 = n_j\alpha$

$$\begin{aligned}
\beta_j k_j(y) &\leq \int_{-\infty}^{\infty} e^{p_j(\alpha_2^2 - t^2)} \left|\varphi_j(\alpha_2 + it)\right| \left|\mathcal{J}_j(y, \alpha_2 + it)\right| \, |\alpha_2 + it| \, dt \\
&\ll e^{p_j\alpha_2^2} |y|^{\alpha_2} \int_0^{\infty} e^{-p_j t^2 + \frac{1}{2}(1-n_j)\pi t} (1+t)^{m_j - 2\alpha_2 - \frac{1}{2}n_j + 1} \, dt \\
&\ll p_j^{-1/2} e^{p_j n_j^2 \alpha^2} |y|^{n_j\alpha} \int_0^{\infty} e^{-t^2} \left(1 + t/\sqrt{p_j}\right)^{m_j - (2\alpha + 1/2)n_j + 1} dt \\
&= p_j^{(\alpha+1/4)n_j - m_j/2 - 1} |y|^{n_j\alpha} \int_0^{\infty} e^{-t^2} \left(\sqrt{p_j} + t\right)^{m_j - (2\alpha+1/2)n_j + 1} dt \\
&\ll \begin{cases} p_j^{-1/2} e^{p_j n_j^2 \alpha^2} |y|^{n_j\alpha} & \text{as } p_j \to \infty\,, \\ p_j^{(\alpha+1/4)n_j - m_j/2 - 1} |y|^{n_j\alpha} & \text{as } p_j \downarrow 0\,. \end{cases}
\end{aligned}$$

For $|y| \geq 1$ we treat the real and complex places separately. At the real places we need only look at $y \geq 1$. (We apply the sum formula with $r = r'$.) For $y \geq 1$ and $j \leq d$ we find from (17) and (10):

$$\begin{aligned}
\beta_j k_j(y) &\ll e^{p_j\alpha_2^2} \int_{-\infty}^{\infty} e^{-p_j t^2} \left(1 + |t|\right)^{m_j+1} dt \\
&\ll \begin{cases} p_j^{-1/2} e^{p_j n_j^2 \alpha^2} & \text{as } p_j \to \infty\,, \\ p^{-m_j/2-1} & \text{as } p_j \downarrow 0\,. \end{cases}
\end{aligned}$$

For $|y| \geq 1$ and $j > d$ we obtain from (16) and (15):

$$\begin{aligned}\beta_j k_j(y) &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-p_j t^2}\varphi_j(it)\mathcal{B}_j(y,it)\, t\sinh \pi t\, dt\\ &\ll |y|^{-1/4}\int_0^{\infty} e^{-p_j t^2}\,(1+t)^{m_j+1}\; t\, dt\\ &\ll \begin{cases} p_j^{-1}|y|^{-1/4} & \text{as } p_j \to \infty\,,\\ p_j^{-m_j/2-3/2}|y|^{-1/4} & \text{as } p_j \downarrow 0\,.\end{cases}\end{aligned}$$

For the total Bessel transform we obtain

$$Bk(y) \ll \begin{cases} \prod_{j=1}^{d+e} p_j^{-1/2} e^{p_j n_j^2\alpha^2}\min\left(|y|^{n_j\alpha},1\right) & \text{as } p_j\to\infty\,,\\ \prod_{j=1}^{d+e} p_j^{-(m_j+n_j+1)/2}\min\left(|y_j|^{n_j\alpha},1\right) & \text{as } p_j \downarrow 0\,.\end{cases}$$

This implies $Bk(y) \ll Q(p)\prod_{j=1}^{d+e}\min\left(|y_j|^{n_j\alpha},1\right)$, with $Q(p) = \prod_{j=1}^{\infty} Q_j(p_j)$, and

$$Q_j(p_j) = \begin{cases} p_j^{-1/2}e^{p_j n_j^2\alpha^2} & \text{if } p_j \geq 1\,,\\ p_j^{-(m_j+n_j+1)/2} & \text{if } 0 < p_j < 1\,.\end{cases}$$

Kloosterman term. Let $\delta \in \left(0,\frac{1}{2}\right)$. We use the Salié-Weil type estimate (7) of Kloosterman sums to obtain

$$K_{-r,-r}(Bk) \ll \sum_{c\in I\smallsetminus\{0\}} |N(c)|^{-1/2+\delta}\, Q(p)\prod_{j=1}^{d+e}\min\left(|(r/c)^{\sigma_j}|^{2n_j\alpha},1\right).$$

For a given $c \in I\smallsetminus\{0\}$, we apply Lemma 8.1 in the next section, with $a = 0$ and $b = 2\alpha$,

$$\begin{aligned}&\sum_{\varepsilon\in\mathcal{O}^*}\prod_{j=1}^{d+e}\min\left(|(r/c\varepsilon)^{\sigma_j}|^{2n_j\alpha},1\right)\\ &\qquad\ll \left(1+\left|\log|N(c/r)|\right|^{d+e-1}\right)\min\left(|N(c/r)|^{0},|N(c/r)|^{-2\alpha}\right)\\ &\qquad\ll_{r,\delta} \min\left(|N(c)|^{-2\alpha+\delta},|N(c)|^{-\delta}\right).\end{aligned}$$

The norm $N(J)$ of the ideal J generated by c is equal to $|N(c)|$. We are left with a sum over the non-zero ideals contained in I,

$$K_{-r,-r}(Bk) \ll Q(p)\sum_J N(J)^{-1/2+\delta}\min\left(N(J)^{-2\alpha+\delta},N(J)^{-\delta}\right).$$

The number of ideals in $\mathcal{O}$ with norm n is $\mathrm{O}(n^\delta)$, see Lemma 4.2 in Chap. 4, of [N, p. 12]. Thus we get

$$K_{-r,-r}(Bk) \ll Q(p)\sum_{n=1}^{\infty} n^{2\delta-1/2}\min(n^{-2\alpha+\delta},n^{-\delta})\,.$$

We have fixed $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$. So the sum converges if we take δ small enough. Hence $K_{-r,-r}(Bk) \ll Q(p)$.

Delta term. We estimate the j-th factor of the integral $\int_Y k(\nu)\, d\eta(\nu)$ by $\int_0^\infty e^{-p_j t^2}\,(1+t)^{m_j}\; t\,dt \ll Q_j(p_j)$. This gives $\Delta_{r,r}(k) \ll Q(p)$.

Spectral term. The sum formula (18) implies that $\int_Y k(\nu)\, d\sigma_{r,r}(\nu) \ll Q(p)$ as well.

Choice of the parameters. Take a subset $E \subset \{1, \ldots, d+e\}$. We take $p_j = t \geq 1$ for $j \in E$, and $p_j = u \in (0,1)$ for $j \notin E$. So

$$\int_Y k(\nu)\, d\sigma_{r,r}(\nu) \ll Q(p) = u^{-g/2-m/2} t^{-f/2} e^{tn\alpha^2}\,,$$

with the notation $f := |E|$, $g := \sum_{j\notin E}(n_j+1)$, $n := \sum_{j\in E} n_j^2$, and $m := \sum_{j\notin E} m_j$.

Support of the measure. The function $\Phi := \times_{j=1}^{d+e}\varphi_j$ is strictly positive on Y, so the measures $d\sigma_{r,r}$ and $\Phi\, d\sigma_{r,r}$ have the same support.

For each $X > 0$, we define the following subset of Y:

$$Y_E(\alpha, X) := \left\{ \nu \in Y : \begin{cases} \nu_j \in [n_j\alpha, n_j/2) & \text{if } j \in E\,, \\ \nu_j \in i[0,X) \cup (0, n_j/2) & \text{otherwise} \end{cases} \right\}.$$

As $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$ has been chosen arbitrarily, to prove the claim it suffices to show that $Y_E(\alpha, X) \cap \operatorname{supp}(\Phi\, d\sigma_{r,r}) = \emptyset$ for all $X > 0$, if $E \neq \emptyset$.

The positive number $\prod_{j\in E} e^{tn_j^2\alpha^2} \prod_{j\notin E} e^{-uX^2} = e^{tn\alpha^2 - huX^2}$, with $h := d+e-f$, is a lower bound of the values of the function $\nu \mapsto \prod_{j=1}^{d+e} e^{p_j\nu_j^2}$ on $Y_E(\alpha, X)$. The measure $\Phi\, d\sigma_{r,r}$ is non-negative. Hence

$$\begin{aligned}\int_{Y_E(\alpha,X)} \Phi(\nu)\, d\sigma_{r,r}(\nu) &\leq e^{-tn\alpha^2 + huX^2} \int_Y k(\nu)\, d\sigma_{r,r}(\nu) \\ &\ll t^{-f/2} u^{-g/2-m/2} e^{huX^2}\,. \end{aligned} \tag{20}$$

This holds for all $t \geq 1$ and all $u \in (0,1)$, but $\int_{Y_E(\alpha,X)} \Phi(\nu)\, d\sigma_{r,r}(\nu)$ does not depend on t and u.

Let $E \neq \emptyset$. So $f > 0$, and the limit as $t \to \infty$ of the right-hand side in (20) is zero. This shows that $Y_E(\alpha, X)$ does not meet $\operatorname{supp}(\Phi\, d\sigma_{r,r}) = \operatorname{supp}(d\sigma_{r,r})$, as asserted.

Estimation of the measure. If we take $E = \emptyset$ and $u = X^{-2}$, then (20) implies

$$\int_{Y(X)} \Phi(\nu)\, d\sigma_{r,r}(\nu) \ll X^{2d+3e+\sum m_j} \qquad \text{as } X \to \infty\,. \tag{21}$$

This contains estimate (19) in the proposition as a special case. □

8 Sum Over the Units

In the present situation, the quotient $\Gamma_N\backslash\Gamma_P$ is in general infinite. This adds an additional difficulty in convergence arguments in the proof of the sum formula and in applications, in comparison with the case of real rank one. In this section we give a lemma useful to handle this difficulty.

LEMMA 8.1. *Let $a,b\in\mathbb{R}$, $a+b>0$. There exists $C\geq 0$ such that for all $f:(\mathbb{R}^*)^d\times(\mathbb{C}^*)^e\to\mathbb{C}$ satisfying*

$$|f(y)|\leq\prod_{j=1}^{d+e}\min\left(|y_j|^{n_j a},|y_j|^{-n_j b}\right),$$

we have

$$\sum_{\varepsilon\in\mathcal{O}^*}|f(\varepsilon y)|\leq C\bigl(1+\bigl|\log|N(y)|\bigr|^{d+e-1}\bigr)\min\left(|N(y)|^a,|N(y)|^{-b}\right).$$

REMARK. We have defined εy by $(\varepsilon y)_j=\varepsilon^{\sigma_j}y_j$.

Proof. The map $\varepsilon\mapsto(\log|\varepsilon^{\sigma_1}|,\dots,\log|\varepsilon^{\sigma_d}|,2\log|\varepsilon^{\sigma_{d+1}}|,\dots,2\log|\varepsilon^{\sigma_{d+e}}|)$ from $\mathcal{O}^*$ to $\mathbb{R}^{d+e}$ has a finite kernel (consisting of the roots of unity in F). Its image is a lattice Λ of dimension $d+e-1$ in $\mathbb{R}^{d+e}$, contained in the hyperplane $T(x):=x_1+\cdots+x_{d+e}=0$. (See, e.g., [L, Chap. V, p. 104].) We write $v_j=n_j\log|y_j|$. Then

$$\sum_{\varepsilon\in\mathcal{O}^*}|f(y\varepsilon)|\ll\sum_{\lambda\in\Lambda}e^{\sum_j\varphi(v_j+\lambda_j)},$$

with $\varphi(x):=\min(ax,-bx)$.

The variation of the function $x\mapsto\sum_j\varphi(x_j)$ on a fundamental region of Λ in the hyperplane $T=0$ is bounded by a multiplicative factor that does not depend on the fundamental region. Hence we have

$$\sum_{\lambda\in\Lambda}e^{\sum_j\varphi(v_j+\lambda_j)}\ll\int_{T(x)=0}e^{\sum_j\varphi(v_j+x_j)}\,d_1x\,,$$

where the measure d_1x is the natural one on $T(x)=0$, corresponding to the Euclidean metric. When we go over to another measure that is invariant under translations, this gives a constant factor in the integral. Let us take the measure $dx_1\,dx_2\,\cdots\,dx_{d+e-1}$, and use $x_{d+e}=-x_1-\cdots-x_{d+e-1}$. After a change of variables we obtain an estimate by

$$\int_{x_1=-\infty}^{\infty}\cdots\int_{x_{d+e-1}=-\infty}^{\infty}e^{\varphi(x_1)+\cdots+\varphi(x_{d+e-1})}\cdot e^{\varphi(T(v)-x_1-\cdots-x_{d+e-1})}dx_{d+e-1}\cdots dx_1.$$

A computation shows that for each $k \in \mathbb{N}$

$$\int_{-\infty}^{\infty} \bigl(1+|x-p|^k\bigr) e^{\varphi(x)+\varphi(p-x)}\, dx \ll_{a+b,k} \bigl(1+|p|^{k+1}\bigr) e^{\varphi(p)} .$$

We apply this with $k=0$ to obtain the estimate

$$\bigl(1+|T(v)-x_1-\cdots-x_{d+e-2}|\bigr) e^{\varphi(T(v)-x_1-\cdots-x_{d+e-2})}$$

for the integral over x_{d+e-1}. Proceeding in this way, each integration adds 1 to the exponent of $|T(v)-x_1-\cdots-x_l|$. When all integrations have been carried out, the estimate is $(1+|T(v)|^{d+e-1}) e^{\varphi(T(v))}$. This gives the statement in the lemma. □

9 Auxiliary Test Functions

Definition 9.1. For $\sigma>0$ and $a=(a_1,\ldots,a_{d+e})$, with $a_j > 2+\frac{1}{2}n_j$ for $j=1,\ldots,d+e$, we define $\mathcal{H}_a(\sigma)$ as the set of functions $h=\times_{j=1}^{d+e} h_j$, where each h_j is an even holomorphic function on $|\operatorname{Re}\nu| \le n_j\sigma$ satisfying

$$h_j(\nu) \ll e^{-\frac{1}{2}\pi|\operatorname{Im}\nu|}\bigl(1+|\operatorname{Im}\nu|\bigr)^{-a_j} .$$

Definition 9.2. For $r \in \mathcal{O}' \smallsetminus \{0\}$, and $h \in \mathcal{H}_a(\sigma)$, we define the function $\mathsf{K}_r h$ on G by

$$\mathsf{K}_r h(g) := a_r^{\rho} \int_Y h(\nu) \mathsf{W}_{\nu,r}(g)\, d\eta(\nu) .$$

See (8) for the measure $d\eta$, and (3) for $\mathsf{W}_{\nu,r}$. Recall also that $a_r = a[\xi_r]$, with $\xi_{r,j} = 2\pi n_j |r^{\sigma_j}|$. To see that the integral converges, we write $\mathsf{K}_r h(a[y]) = \times_{j=1}^{d+e} \kappa_{r,j} h_j(y_j)$, with

$$\kappa_{r,j} h_j(y) := \xi_{r,j}^{n_j/2} \frac{i}{2} \int_{\operatorname{Re}\nu=0} h_j(\nu) y^{n_j/2} K_\nu(\xi_{r,j} y)\, \nu \sin \pi\nu\, d\nu . \tag{22}$$

This implies that $\kappa_{r,j} h_j(y) = 2\pi \xi_{r,j}^{n_j/2} y^{n_j/2} \beta_{.} k_{.}\bigl(-\frac{1}{4} n_j^2 |r^{\sigma_j}|^2 y^2\bigr)$, with $k_{.}(\nu) = h_j(2\nu)$, and $\beta_{.}$ as occurs in Lemma 5.8 for the real places. In particular, the integral in (22) converges, and we can move the line of integration in (22) to $\operatorname{Re}\nu = \alpha_1$ for any α_1 satisfying $|\alpha_1| \le n_j\sigma$ and $|\alpha_1| < a_j - \frac{3}{2}$.

Lemma 9.3. *The function $\mathsf{K}_r(h)$ on G/K transforms on the left according to the character χ_r of N, and satisfies*

$$\mathsf{K}_r h(a[y]) \ll \prod_{j=1}^{d+e} \min\bigl(y_j^{(\sigma+1/2)n_j}, y_j^{(-\alpha+1/2)n_j}\bigr)[0] \tag{23}$$

$$\text{with } 0 < \alpha < (a_j - 2)/n_j \text{ for all } j, \text{ and } \alpha \le \sigma,$$

$$\ll \prod_{j=1}^{d+e} \min\left(y_j^{(\sigma+1/2)n_j}, y_j^{(-\alpha+1/2)n_j - 1}\right) \tag{24}$$

$$\text{with } -\frac{1}{2n_j} < \alpha < \frac{a_j - 3}{n_j} \text{ for all } j\,, \text{ and } |\alpha| \le \sigma\,.$$

For $h, h' \in \mathcal{H}_a(\sigma)$,

$$\int_A a^{-2\rho} \mathsf{K}_r h(a)\, \overline{\mathsf{K}_r h'(a)}\, da = \left(\tfrac{\pi^2}{2}\right)^{d+e} a_r^{2\rho} \int_Y h(\nu)\, \overline{h'(\nu)}\, d\eta(\nu)\,. \tag{25}$$

If $\nu \in \mathbb{C}^{d+e}$ satisfies $|\operatorname{Re}\nu_j| < n_j\sigma$ for $j = 1, \ldots, d+e$, then

$$\int_A a^{-2\rho}\, \mathsf{K}_r h(a)\, \mathsf{W}_{\nu,r}(a)\, da = \left(\tfrac{\pi^2}{2}\right)^{d+e} a_r^{\rho} h(\nu)\,. \tag{26}$$

Proof. The behavior under N and K is clear. For large y_j in (23), we move the line of integration in (22) to $\operatorname{Re}\nu = n_j\alpha$, and use (41). In (24) we proceed similarly, with use of (42).

For small y_j, we rewrite (22) with the help of (36),

$$\kappa_{r,j} h_j(y) = -\frac{\pi i}{2} \xi_{r,j}^{n_j/2} y^{n_j/2} \int_{\operatorname{Re}\nu = 0} h_j(\nu) I_\nu(\xi_{r,j} y)\, \nu\, d\nu\,, \tag{27}$$

move the line of integration to $\operatorname{Re}\nu = n_j\sigma$, and use (38).

For the other statements we use results from §14 of [B]. (Three-number references in this proof correspond to [B].) For $h \in \mathcal{H}_a(\sigma)$ the function $\varphi : s \mapsto \cos\pi s\, h_j(2s)$ is an element of the space ${}_{-1}F_{0,n_j\sigma/2}^{a_j}$ in Definition 14.2.7. Proposition 14.2.8 implies that $\kappa_{r,j} h_j(y) = 2\pi(\xi_{r,j} y)^{n_j/2} (b_0^{-1})^{\leftarrow} \varphi(\xi_{r,j} y)$. Let $f, f_1 \in C_c^\infty(0,\infty)$. Definition 14.2.1 and Lemma 14.2.3 show that

$$\kappa_{r,j}^{\leftarrow} f(\nu) := \frac{2}{\pi^2} \xi_{r,j}^{-n_j/2} \int_0^\infty y^{-n_j/2} f(y) K_\nu(\xi_{r,j} y)\, \frac{dy}{y}$$

defines $\kappa_{r,j}^{\leftarrow} f \in \mathcal{H}_a(\sigma)$. Proposition 14.2.8 implies that $\kappa_{r,j}(\kappa_{r,j}^{\leftarrow} f) = f$, and gives, together with Proposition 14.2.6:

$$\int_0^\infty y^{-n_j} f(y)\, \overline{f_1(y)}\, \frac{dy}{y} = \frac{\pi^2}{2} \xi_{r,j}^{n_j} \frac{i}{2} \int_{\operatorname{Re}\nu = 0} \kappa_{r,j}^{\leftarrow} f(\nu)\, \overline{\kappa_{r,j}^{\leftarrow} f_1(\nu)}\, \nu \sin\pi\nu\, d\nu\,.$$

This relation extends to f and f_1 of the form $\kappa_{r,j} h_j$ and $\kappa_{r,j} h_j'$, and thus gives (25). After that, we conclude that $\kappa_{r,j}^{\leftarrow} f$ makes sense for $f = \kappa_{r,j} h_j$, and it equals $h_j(\nu)$ for $\operatorname{Re}\nu = 0$. By holomorphy this extends to $|\operatorname{Re}\nu| < n_j\sigma$. □

REMARK. In the proof we have also seen that for each $f \in C_c^\infty(A)$ there exists $h \in \mathcal{H}_a(\sigma)$ with $f = \mathsf{K}_r h$. These f are the auxiliary test functions ψ in [WM, §6].

Intertwining operators. Put $s_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbf{G}_\mathbb{Q} \subset G$. Let $r, r' \in \mathcal{O}' \setminus \{0\}$.

If the function f on G satisfies $f(nak) = \chi_{r'}(n) f(a)$ for $n \in N$, $a \in A$, $k \in K$, and $f(a[y]) \ll N(y)^{1/2+\delta}$ as $N(y) \downarrow 0$ for some $\delta > 0$, then

$$V(t)f(g) := \int_N \chi_r(n)^{-1} f\big(b[t]s_0 n g\big)\, dn \tag{28}$$

converges absolutely for each $t \in (\mathbb{R}^*)^d \times (\mathbb{C}^*)^e$, $g \in G$, and $V(t)f(ngk) = \chi_r(n)\, V(t)f(g)$ for all $n \in N$, $g \in G$, $k \in K$. The study of this integral is essential to obtain Fourier coefficients of Poincaré series, and the Kloosterman term in the sum formula.

PROPOSITION 9.4. *Let $\sigma \geq \sigma' > 0$. Take $t \in (\mathbb{R}^*)^d \times (\mathbb{C}^*)^e$, and $h \in \mathcal{H}_a(\sigma)$, with $a \in \mathbb{R}_{>0}^{d+e}$ such that $a_j > 2 + 2\sigma'$ and $a_j > 2 + \frac{1}{2} n_j$ for $j = 1, \ldots, d+e$. Define $h_t(\nu) := |N(t)| \mathcal{B}(rr't^2, \nu) h(\nu)$.*

Then $h_t \in \mathcal{H}_{a'}(\sigma')$ with $a'_j = a_j - 2\sigma' + \frac{1}{2} n_j$ for $j = 1, \ldots, d+e$, and

$$V(t) K_{r'} h = \pi^e a_r^{-\rho} a_{r'}^{\rho} \mathsf{K}_r h_t \,.$$

REMARK. See Definition 5.5 for $\mathcal{B}$. Note that $(rr't^2)_j = (rr')^{\sigma_j} t_j^2$.

This result is essentially equation (2.10) in [MW1]. The notation and normalizations here differ from those in [MW1] and [MW2]. Hence we give the proof in some detail.

Proof. Estimate (12) implies that $h_t \in \mathcal{H}_{a'}(\sigma')$.

It suffices to show that $V(t)\mathsf{K}_{r'} h(a) = \pi^e a_r^{-\rho} a_{r'}^{\rho} \mathsf{K}_r h_t(a)$ for $a \in A$. This can be considered locally. We introduce

$$f'_j\big(n a_j[y] k\big) = \chi_{r',j}(n) \kappa_{r',j} h_j(y)\,,$$
$$f_{t,j}\big(n a_j[y] k\big) = \chi_{r,j}(n) \kappa_{r,j} h_{t,j}(y)\,.$$

For a fixed t we have to show for each $a \in A_j$,

$$\int_{N_j} \chi_{r,j}(n)^{-1} f'_j\big(b_j[t_j] s_0 n a\big)\, dn = \pi^{n_j - 1} \xi_{r,j}^{-n_j/2} \xi_{r',j}^{n_j/2} f_{t,j}(a)\,, \tag{29}$$

where N_j is the j-th factor of N, similarly for b_j and A_j.

We express f'_j as an integral with an I-Bessel function. We take (27), and move the line of integration to $\mathrm{Re}\,\nu = n_j \sigma'$,

$$f'_j(g) = -\frac{\pi i}{2} \int_{\mathrm{Re}\,\nu = n_j\sigma'} h_j(\nu) M_{r'}(g, \nu)\, \nu\, d\nu\,,$$

where $M_{r'}(na_j[y]k,\nu) := \chi_{r',j}(n)(\xi_{r',j}y)^{n_j/2} I_\nu(\xi_{r',j}y)$.

For $f_{t,j}$ we keep to a K-Bessel function. We obtain from (22) and Definition 5.5,

$$f_{t,j}(g) = -i \int_{\mathrm{Re}\,\nu = n_j\sigma'} h_j(\nu)|t_j|^{n_j} \mathcal{J}_j\big((rr')^{\sigma_j} t_j^2, \nu\big) W_r(g,\nu)\,\nu\,d\nu\,,$$

where $W_r(na_j[y]k,\nu) := \chi_{r,j}(n)(\xi_{r,j}y)^{n_j/2} K_\nu(\xi_{r,j}y)$.

We insert these expressions into (29), and interchange the order of integration. We conclude that it suffices to prove the following equality,

$$\begin{aligned}\frac{\pi}{2}\int_{N_j} &\chi_{r,j}(n)^{-1} M_{r'}\big(b_j[t_j]s_0 na,\nu\big)\,dn \\ &= \pi^{n_j-1}\xi_{r,j}^{-n_j/2}\xi_{r',j}^{n_j/2}|t_j|^{n_j}\mathcal{J}_j\big((rr')^{\sigma_j}t_j^2,\nu\big)W_r\,(a,\nu)\ . \qquad (30)\end{aligned}$$

Now we use results in [MW1] and [MW2]. Let $v_0 : na_j[y]k \mapsto y^{\nu+n_j/2}$. This is the standard K-invariant vector in the induced representation $H^\nu = H^{1,\nu}$ as in [MW2, §1]. The function $y \mapsto M_{r'}(a_j[y],\nu)$ is asymptotic to a multiple of $y \mapsto y^{\nu+n_j/2}$ as $y \downarrow 0$. So $M_{r'}(g,\nu)$ should be a multiple of $M(1,\nu,g,v_0) = M^{\chi_{r',j}}(1,\nu,g,v_0)$. The normalization in [MW2] and the characterization in Theorem 1.1 of [MW1] imply that $M_{r'}(g,\nu) = 2^{-\nu}\,\Gamma\,(\nu+1)^{-1}\,\xi_{r',j}^{\nu+n_j/2} M^{\chi_{r',j}}(1,\nu,g,v_0)$. In the notation of [MW1, §1]:

$$M_{r'}(g,\nu) = 2^{-\nu}\,\Gamma\,(\nu+1)^{-1}\,\xi_{r',j}^{\nu+n_j/2}\hat{I}_1(\nu)^{-1}\varpi(1,\nu)\big(\pi_{1,\nu}(g)v_0\big)\,.$$

Here $\hat{I}_1$ is the function denoted by I_ξ (for $\xi = 1$) in [MW1].

In (1.4) of [MW2] we find an expression for the Jacquet Whittaker vector applied to v_0. It gives $J^{\chi_{r,j}}(\nu,g) = d_{r,j}(\nu)\xi_{r,j}^{-n_j/2}W_r(g,\nu)$. (The d's, see (4), can be found in [MW2, p. 193–194].)

Formula (2.10) in [MW1] shows that the left-hand side of (30) is equal to

$$\frac{\pi\xi_{r',j}^{n_j/2+\nu}}{2^{1+\nu}\Gamma(1+\nu)}|t_j|^{2\nu+n_j}\frac{\tau(\chi_{r,j},\chi_{r',j},b_j[t_j],\nu)}{\hat{I}_1(\nu)}d_{r,j}(\nu)\xi_{r,j}^{-n_j/2}W_r(a,\nu)\,,$$

for a function $\tau(\chi_{r,j},\chi_{r',j},b_j[t_j],\nu)$ that is meromorphic in ν. (We have used $1_\nu\,(b_j[t_j]) = |t_j|^{2\nu+n_j}$, see line 12 on p. 438 of [MW1].)

In the case $j \le d$ we find on p. 461 of [MW1] that

$$\begin{aligned}\tau\big(&\chi_{r,j},\chi_{r',j},b_j[t_j],\nu\big)\\ &= \Big(2\pi\sqrt{|(rr')^{\sigma_j}|}|t_j|\Big)^{-2\nu}J_{2\nu}^{\mathrm{sign}((rr')^{\sigma_j})}\Big(4\pi\sqrt{|(rr')^{\sigma_j}|}|t_j|\Big)\\ &= \xi_{r,j}^{-\nu}\xi_{r',j}^{-\nu}|t_j|^{-2\nu}\mathcal{J}_j\big((rr')^{\sigma_j}t_j^2,\nu\big)\,,\end{aligned}$$

and on p. 462 we see that $\hat{I}_1(\nu) = \Gamma(2\nu+1)^{-1}$. This gives (30) in the real case.

In the complex case, $j > d$, we find an expression for τ after Proposition A.6 in [MW2]. It should be divided by $\hat{I}_1(\nu)$, due to a difference in normalization between [MW1] and [MW2]. (There is also a slight mistake in the factor $\Gamma(\nu+1)^2$: the exponent is missing in [MW1].) This gives the following equation, and yields (30) for $j > d$.

$$\frac{\tau(\chi_{r,j}, \chi_{r',j}, b_j[t_j], \nu)}{\hat{I}_1(\nu)} = \Gamma(\nu+1)^2 2^{2\nu} \xi_{r,j}^{-\nu} \xi_{r',j}^{-\nu} |t_j|^{-2\nu} \mathcal{J}_j(r_j r_j' t_j^2, \nu).$$

$\square$

10 Proof of the Sum Formula

In this section we prove Theorem 6.1.

Discussion of the approach. In the case of a group of real rank one, the sum formula ([MW2, Theorem 1.9]) is first proved for a restricted class of test functions and then extended to a wider class, by means of an approximation argument, and with help of the density result in Lemma 1.7 of [MW2]. But in that proof one uses the boundedness on vertical strips of the quantity $D^{\chi}_{\chi'}(\nu)$, see [MW2, Proposition 1.2 (v)]. This function $D^{\chi}_{\chi'}$ is a Fourier coefficient of a Poincaré series with exponential growth, as studied in [MW1]. In the present situation, the fact that $\Gamma_N \backslash \Gamma_P$ can be infinite prevents the construction of this type of Poincaré series by a sum over $\Gamma_N \backslash \Gamma$.

Thus, our approach will be to adapt the proof in [WM, §6] to the present situation. That proof is based on the scalar product of two bounded Poincaré series, and the choice made in [WM] is to start with compactly supported functions on A. In that way the convergence problems due to the presence of infinitely many units are avoided; but the drawback is that the class of test functions is too restricted for our purposes.

In the sequel we follow closely the proof in §6 of [WM].

Test functions. Let σ and a be as in Theorem 6.1. We write the principal test function $k \in \mathcal{K}_a(\sigma)$ as $k = h\overline{h'}$, with auxiliary test functions $h, h' \in \mathcal{H}_{a/2}(\sigma)$. (We use $\overline{h'}(\nu) = \overline{h'(\bar{\nu})}$.) This can be arranged by taking

$$h'(\nu) := \prod_{j=1}^{d+e} \left((p-\nu^2)^{-a_j/4-1} \frac{1-\nu^2}{\cos\frac{\pi\nu}{2}} \right), \qquad h := k/\overline{h'} = k/h',$$

with $p > 4\sigma^2$.

To keep close to the notation in [WM] we put $\psi(a) := \mathsf{K}_r h(a)$, $\psi'(a) := \mathsf{K}_{r'} h'(a)$. Here these functions are not necessarily compactly supported.

Poincaré series. As in [WM, p. 314] we set

$$P_\psi^r(g) := \sum_{\gamma\in\Gamma_N\backslash\Gamma} \chi_r(n(\gamma g))\psi(a(\gamma g))\,,$$

and similarly for $P_{\psi'}^{r'}$. (We use $g = n(g)a(g)k(g)$ corresponding to $G = NAK$.) Note that P_ψ^r corresponds to P_ψ^χ in [WM].

We first check the absolute convergence of the series. From (23) we have $\psi(a[y]) \ll \prod_{j=1}^{d+e} \min(y_j^{n_j(\sigma+1/2)}, y_j^{n_j(-\alpha+1/2)})$, for some $\alpha \in (1/2, \sigma)$. We write

$$\sum_{\gamma\in\Gamma_N\backslash\Gamma} \bigl|\chi_r(n(\gamma g)\psi(a(\gamma g))\bigr| = \sum_{\gamma\in\Gamma_P\backslash\Gamma}\sum_{\delta\in\Gamma_N\backslash\Gamma_P} \bigl|\psi(a(\delta)a(\gamma g)\bigr|\,.$$

For the inner sum we use

$$\begin{aligned}\sum_{\delta\in\Gamma_N\backslash\Gamma_P} \bigl|\psi(a(\delta)a[y])\bigr| &= \sum_{\varepsilon\in\mathcal{O}^*} \bigl|\psi(b[\varepsilon]a[y])\bigr| = \sum_{\varepsilon\in\mathcal{O}^*} \bigl|\psi(a[|\varepsilon|^2 y])\bigr| \\ &\le \sum_{\varepsilon\in\mathcal{O}^*} \bigl|\psi(a[|\varepsilon| y])\bigr|\,.\end{aligned}$$

(Interpretation: $(|\varepsilon|^2 y)_j = |\varepsilon^{\sigma_j}|^2 y_j$.) Lemma 8.1 shows that this sum has the estimate $\mathrm{O}(\min(N(y)^{\sigma+(1-\delta)/2}, N(y)^{-\alpha+(1+\delta)/2}))$ for each $\delta > 0$. So $\sum_{\gamma\in\Gamma_N\backslash\Gamma_P} |\psi(a(\delta g))| \ll a(g)^{(2\sigma-\delta+1)\rho}$. This means that

$$\sum_{\gamma\in\Gamma_P\backslash\Gamma}\sum_{\delta\in\Gamma_N\backslash\Gamma_P} \bigl|\psi(a(\delta\gamma g))\bigr| \ll E\bigl(P, (2\sigma-\delta)\rho, 0, g\bigr)\,,$$

which converges if δ is small enough.

Moreover, as $N(y)\to\infty$, we have

$$\begin{aligned}&\sum_{\gamma\in\Gamma_P\backslash\Gamma,\,\gamma\notin\Gamma_P}\ \sum_{\delta\in\Gamma_N\backslash\Gamma_P} \bigl|\psi(a(\delta\gamma g))\bigr| \\ &\qquad \ll E\bigl(P, (2\sigma-\delta)\rho, 0, g\bigr) - a(g)^{\rho+(2\sigma-\delta)\rho} = \mathrm{O}(1)\,,\end{aligned}$$

and $\sum_{\delta\in\Gamma_N\backslash\Gamma_P} |\psi(a(\delta g)| \ll a(g)^{(1+\delta-2\alpha)\rho} = \mathrm{O}(1)$, provided we take $\delta < 2\alpha - 1$. So P_ψ^r is bounded on Siegel domains associated to P. If Q is a percuspidal parabolic group not Γ-conjugate to P, then the Eisenstein series $E\,(P, (2\sigma-\delta)\,\rho, 0, g_1)$ itself is bounded on the corresponding Siegel domain. We conclude that the Poincaré series converges absolutely, and that P_ψ^r is a bounded continuous function on $\Gamma\backslash G/K$. This holds even for the sum of the absolute values of the terms in the Poincaré series.

Condition on the a_j. The convenience to use bounded Poincaré series forces us to take $a_j > 2 + \frac{1}{2}n_j$ in Definition 9.1 of the auxiliary test functions, and to put the condition $a_j > 4 + n_j$ on the main test functions in Theorem 6.1.

Fourier coefficients. The absolute convergence makes the computation of $\langle P^r_\psi, \psi_\ell \rangle$ on p. 314 of [WM] go through. We obtain

$$\langle P^r_\psi, \psi_\ell \rangle = \int_A a^{-2\rho}\, \psi(a)\, \overline{c_r(\psi_\ell)}\, \overline{d_r(\overline{\mu_\ell})}\, \mathsf{W}_{\mu_\ell, r}(a) da\,.$$

We use that the components of μ_ℓ satisfy $\overline{\mu_{\ell,j}} = \pm\mu_{\ell,j}$, and that $\mathsf{W}_{\nu,r}$ is invariant under $\nu_j \mapsto -\nu_j$. Application of equation (26) gives

$$\langle P^r_\psi, \psi_\ell \rangle = \left(\tfrac{\pi^2}{2}\right)^{d+e} \overline{c_r(\psi_\ell)}\, d_r(\mu_\ell)\, a_r^\rho\, h(\mu_\ell)\,.$$

Note that $c_r(\psi_0) = 0$.

As P^r_ψ is bounded, the integration against an Eisenstein series with its spectral parameter on the critical line converges absolutely. Thus, we similarly obtain for $z \in i\mathbb{R}$

$$\left\langle P^r_\psi, E(P^j, z\rho, i\mu) \right\rangle = \left(\tfrac{\pi^2}{2}\right)^{d+e} \overline{D_r(P^j, z\rho, i\mu)}\, d_r(z\rho + i\mu)\, a_r^\rho h(z\rho + i\mu)\,.$$

For $P^{r'}_{\psi'}$, built with the character $\chi_{r'}$ and $\psi' = K_{r'}h'$, we have similar results.

Spectral term. We apply the inner product formula (6), and obtain

$$\langle P^r_\psi, P^{r'}_{\psi'} \rangle = \left(\tfrac{\pi^2}{2}\right)^{2d+2e} a_r^\rho a_{r'}^\rho \int_Y k(\nu)\, d\sigma_{r,r'}(\nu)\,, \tag{31}$$

see Definition 3.4. In particular, k is integrable for $d\sigma_{r,r'}$.

Geometric computation of the scalar product. The scalar product can be computed in another way as well. The resulting equality is the sum formula.

The sum of the absolute values of the terms in the Poincaré series is a bounded function. Hence we have

$$\begin{aligned}\langle P^r_\psi, P^{r'}_{\psi'} \rangle &= \int_A a^{-2\rho} \psi(a) \int_{\Gamma_N \backslash N} \chi_r(n) \overline{P^{r'}_{\psi'}(na)}\, dn\, da \\ &= \sum_{\gamma \in \Gamma_N \backslash \Gamma} \int_A a^{-2\rho} \psi(a) \int_{\Gamma_N \backslash N} \chi_r(n) \chi_{r'}(n(\gamma n a))^{-1} \overline{\psi'(a(\gamma n a))}\, dn\, da\,.\end{aligned}$$

This converges absolutely, so we can evaluate the sums and integrals in any order that suits us. As in [WM] we first consider the sum I_1 over $\gamma \in$

$\Gamma_N\backslash\Gamma_P$, which will yield the delta term of the sum formula. Next we shall obtain the Kloosterman term from the sum I_2 over $\Gamma_N\backslash(\Gamma\smallsetminus\Gamma_P)$. This decomposition corresponds to the two cells of the Bruhat decomposition of $\mathbf{G}_{\mathbb{Q}}\cong\mathrm{SL}_2(F)$.

The delta term. We consider first

$$\begin{aligned}\int_{\Gamma_N\backslash N}&\sum_{\gamma\in\Gamma_N\backslash\Gamma_P}\chi_r(n)\chi_{r'}(n(\gamma na))^{-1}\overline{\psi'(a(\gamma na))}\,dn\\&=\sum_{\gamma\in\Gamma_N\backslash\Gamma_P}\overline{\psi'(a(\gamma a))}\int_{\Gamma_N\backslash N}\chi_r(n)\chi_{r'}(\gamma n\gamma^{-1})^{-1}\,dn\,.\end{aligned}$$

Let us take the representatives $b[\varepsilon]=\left(\begin{smallmatrix}\varepsilon&0\\0&1/\varepsilon\end{smallmatrix}\right)$, $\varepsilon\in\mathcal{O}^*$, of $\Gamma_N\backslash\Gamma_P$. The integral over n is non-zero solely if $r=\varepsilon^2r'$. We obtain

$$I_1=\sum_{\varepsilon\in\mathcal{O}^*,\,r=\varepsilon^2r'}\mathrm{vol}(\Gamma_N\backslash N)\int_A a^{-2\rho}\psi(a)\overline{\psi'\bigl(a(b[\varepsilon]a)\bigr)}\,da\,.$$

If $\varepsilon\in\mathcal{O}^*$, then $a\left(b[\varepsilon]a[y]\right)=a[|\varepsilon|^2y]$ with $(|\varepsilon|^2y)_j=|\varepsilon^{\sigma_j}|^2y_j$. If moreover $r=\varepsilon^2r'$, then $\xi_{r,j}=|\varepsilon^{\sigma_j}|^2\xi_{r',j}$, and $a_{r'}^{\rho}=a_r^{\rho}$. Hence $\psi'(a(b[\varepsilon]a))=\mathsf{K}_{r'}h'(a(b[\varepsilon]a))=\mathsf{K}_rh'(a)$. We use equation (25), and Definition 5.3, to conclude that $I_1=\alpha(r,r')\bigl(\frac{\pi^2}{2}\bigr)^{2d+2e}a_r^{2\rho}\int_Y k(\nu)\,d\eta(\nu)$. We have obtained

$$I_1=\left(\frac{\pi^2}{2}\right)^{2d+2e}a_r^{\rho}a_{r'}^{\rho}\,\Delta_{r,r'}(k)\,. \tag{32}$$

Contribution of the big cell. The absolute convergence implies that the other contribution to the scalar product of P_ψ^r and $P_{\psi'}^{r'}$ is equal to

$$I_2=\sum_{\xi\in\Gamma_N\backslash(\Gamma\smallsetminus\Gamma_P)/\Gamma_N}\int_A a^{-2\rho}\psi(a)\int_N\chi_r(n)\overline{K_{r'}h'(\xi na)}\,dn\,da\,,$$

corresponding to equation (18) in [WM], but without the integral representation for ψ'. For our Γ, a system of representatives of $\Gamma_N\backslash(\Gamma\smallsetminus\Gamma_P)/\Gamma_N$ can be chosen as $\mathcal{S}=\bigl\{\xi_{\gamma,\delta}:\gamma\in I,\,\gamma\neq0,\,\delta\in(\mathcal{O}/\gamma\mathcal{O})^*\bigr\}$. So δ runs over representatives of $(\mathcal{O}/\gamma\mathcal{O})^*$ and $\xi_{\gamma,\delta}=\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)\in\Gamma$.

The Bruhat decomposition of $\mathbf{G}_{\mathbb{Q}}\cong\mathrm{SL}_2(F)$ allows us to write $\xi=\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)\in\mathrm{SL}_2(F)$, with $\gamma\neq0$, uniquely in the form $\xi=n_1(\xi)m(\xi)a_\xi s_0n_2(\xi)$, with $n_1(\xi)=n[\alpha/\gamma]$, $n_2(\xi)=n[\delta/\gamma]\in N\cap\mathbf{G}_{\mathbb{Q}}$, $m(\xi)\in M\cap\mathbf{G}_{\mathbb{R}}$, $a_\xi\in A\cap\mathbf{G}_{\mathbb{R}}$, and $s_0=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. We have $a_\xi=a[y_\gamma]$, with $y_{\gamma,j}=|\gamma^{\sigma_j}|^{-2}$, and $a_\xi m(\xi)=m(\xi)a_\xi=b[1/\gamma]$.

For $\xi = \xi_{\gamma,\delta} \in \mathcal{S}$ we have

$$\overline{\int_N \chi_r(n)\overline{\mathsf{K}_{r'}h'(\xi na))}\,dn} = \chi_r(n_2(\xi))\chi_{r'}(n_1(\xi)) \cdot \int_N \chi_r(n)^{-1}\mathsf{K}_{r'}h'\big(b[1/\gamma]s_0na\big)\,dn\,.$$

In (28) and Proposition 9.4, with $0 < \sigma' < \frac{1}{4}n_j$, we see that the integral over N is equal to

$$V(1/\gamma)\mathsf{K}_{r'}h'(a) = \pi^e a_{r'}^\rho a_r^{-\rho}\mathsf{K}_r h'_{1/\gamma}(a)\,.$$

We use (25) with $h, h'_{1/\gamma} \in \mathcal{H}_{a/2}(\sigma')$, Proposition 9.4, and Definition 5.7 to see that

$$\begin{aligned}
&\int_A a^{-2\rho}\mathsf{K}_r h(a)\pi^e a_{r'}^\rho a_r^{-\rho}\overline{K_r h'_{1/\gamma}(a)}\,da\\
&\quad = \pi^e a_{r'}^\rho a_r^{-\rho}\left(\tfrac{\pi^2}{2}\right)^{d+e} a_r^{2\rho}\int_Y h(\nu)\overline{h'_{1/\gamma}(\nu)}\,d\eta(\nu)\\
&\quad = a_r^\rho a_{r'}^\rho 2^{-d-e}\pi^{2d+3e}\int_Y k(\nu)\,|N(\gamma)|^{-1}\,\mathcal{B}(rr'/\gamma^2,\nu)\,d\eta(\nu)\\
&\quad = 2^{-d-e}\pi^{2d+3e}a_r^\rho a_{r'}^\rho\,|N(\gamma)|^{-1}\,Bk(rr'/\gamma^2)\,.
\end{aligned}$$

For $\xi = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, we find $\chi_r(n_2(\xi))\chi_{r'}(n_1(\xi)) = e^{2\pi i S(r\delta/\gamma + r'\alpha/\gamma)}$ (see section 2 for S). Hence

$$\begin{aligned}
I_2 &= 2^{-d-e}\pi^{2d+3e}a_r^\rho a_{r'}^\rho \sum_{\gamma\in I,\,\gamma\neq 0}|N(\gamma)|^{-1}\,Bk(rr'/\gamma^2)\\
&\qquad\cdot\sum_{\delta\in(\mathcal{O}/(\gamma))^*} e^{-2\pi i S(r\delta/\gamma + r'\alpha/\gamma)}\\
&= 2^{-d-e}\pi^{2d+3e}a_r^\rho a_{r'}^\rho K_{-r,-r'}(Bk)\,,
\end{aligned}\tag{33}$$

see Definition 4.2.

We have computed $\langle P_\psi^r, P_{\psi'}^{r'}\rangle$ in two ways, see (31), (32) and (33). All terms are absolutely convergent. We divide the resulting equality by $(\pi^2/2)^{2d+2e}a_r^\rho a_{r'}^\rho$ to obtain (18). $\square$

Remark. Comparison with Theorem 6.1 in [WM] shows a few differences. In the formula for I_1 at the top of p. 317 there is a missing factor $(-i)^t$. The delta term in Theorem 6.1 should be multiplied by $(-1)^t$ (with $t = d+e$). The formula for $\overline{I_2}$ on p. 317 should have a factor $\left(\frac{1}{2\pi i}\right)^t$. This propagates through the proof. The Kloosterman term in Theorem 6.1 should have

$(2\pi)^{-t}a^{\rho}_{\delta}$ as an additional factor. In the computation of all terms in the sum formula in [WM] the factor $a^{\rho}_{\chi}a^{\rho}_{\chi'}$ is missing.

11 Bessel Functions

In this rather technical section, we collect some results concerning Bessel functions that we have used in the previous sections.

$$J^1_\nu(y) := J_\nu(y) = \sum_{n=0}^{\infty} \frac{(-1)^n (y/2)^{\nu+2n}}{\Gamma(\nu+1+n)\, n!} \tag{34}$$

$$J^{-1}_\nu(y) := I_\nu(y) = \sum_{n=0}^{\infty} \frac{(y/2)^{\nu+2n}}{\Gamma(\nu+1+n)\, n!} \tag{35}$$

$$K_\nu(y) = \frac{\pi}{2}\frac{I_{-\nu}(y) - I_\nu(y)}{\sin \pi\nu} \tag{36}$$

$$P_\nu(z) := I_\nu(z) I_\nu(\bar z) = \left|\frac{z}{2}\right|^{2\nu} \sum_{n,m\ge 0} \frac{(z/2)^{2n}(\bar z/2)^{2m}}{\Gamma(\nu+1+n)\,\Gamma(\nu+1+m) n! m!}\,. \tag{37}$$

See, e.g., [Wa, §3.1 and §3.7]. The notation $J^{\pm 1}_\nu$ is unusual, but sometimes convenient. The function P_ν is an even and smooth on $\mathbb{C}^*$.

Lemma 11.1. *Let $y_0 > 0$.*

$$J^{\pm1}_\nu(y) \ll_{y_0} \left|\Gamma(\nu+1)\right|^{-1} y^{\operatorname{Re}\nu} \quad \text{for } 0 < y \le y_0\,, \tag{38}$$

$$P_\nu(y) = I_\nu(y) I_\nu(\bar y) \ll_{y_0} \left|\Gamma(\nu+1)\right|^{-2} |y|^{2\operatorname{Re}\nu} \text{for } 0 < |y| \le y_0\,, \tag{39}$$

$$J_\nu(y) \ll e^{\frac12\pi|\operatorname{Im}\nu|}/\operatorname{Re}\nu \qquad \text{for } y>0\,,\ \operatorname{Re}\nu > 0\,, \tag{40}$$

$$K_\nu(y) \ll \left|\Gamma\left(\nu+\tfrac12\right)\right| y^{-\operatorname{Re}\nu} \qquad \text{for } y>0\,,\ \operatorname{Re}\nu > 0\,, \tag{41}$$

$$\ll \left|\Gamma\left(\nu+\tfrac32\right)\right| y^{-\operatorname{Re}\nu-1} \qquad \text{for } y>0\,,\ \operatorname{Re}\nu > -\tfrac12\,, \tag{42}$$

$$P_{-\nu}(y) - P_\nu(y) \ll \bigl(1+|\nu|\bigr)|y|^{\frac{-1}{2}} e^{\frac32\pi|\nu|} \qquad \text{for } \operatorname{Re}\nu = 0\,,\ |y| \ge 1\,. \tag{43}$$

Proof. (38) and (39) follow directly from the power series expansions. For estimate (40) we use the integral representation (7) in [Wa, 6.21].

Basset's integral, [Wa, 6.16],

$$K_\nu(y) = \tfrac12 \pi^{-1/2}(y/2)^{-\nu}\,\Gamma\left(\nu+\tfrac12\right) \int_{-\infty}^{\infty} e^{-iyu}(u^2+1)^{-\nu-1/2}\,du\,,$$

implies (41). Partial integration gives

$$K_\nu(y) = \tfrac{i}{\sqrt{\pi}}\,\Gamma\left(\nu+\tfrac32\right) 2^\nu y^{-\nu-1} \int_{-\infty}^{\infty} e^{-iyu}(u^2+1)^{-\nu-3/2} u\,du\,,$$

yielding (42).

The proof of (43) takes much more work. We consider $\nu = it$ with $t \geq 0$. We use $I_\nu(y) = J_\nu(iy)$ and the relations in 3.61 of [Wa] to express the left-hand side of (43) in Bessel functions of the third kind,

$$\begin{aligned} &P_{-it}(y) - P_{-it}(y) \\ &\qquad = \tfrac{1}{2}\sinh \pi t\big(e^{\pi t} H_{it}^{(2)}(iy) H_{it}^{(2)}(\,\overline{iy}\,) - e^{-\pi t} H_{it}^{(1)}(iy) H_{it}^{(1)}(\,\overline{iy}\,)\big)\,. \end{aligned}$$

As this is an even function of $y \in \mathbb{C}^*$, we restrict our attention to $\arg y \in [-\pi, 0]$.

For $|z| \geq 1$, $|\arg z| \leq \frac{\pi}{2}$ we find in formulas (2) and (3) in [Wa, 6.51]:

$$\begin{aligned} H_{it}^{(2)}(z) &= \pi^{-5/2} e^{-\pi t - iz} \cosh(\pi t)(2z)^{it} F_{it}(2iz)\,, \\ H_{it}^{(1)}(z) &= -\,\pi^{-5/2} e^{\pi t + iz} \cosh(\pi t)(2z)^{it} F_{it}(-2iz)\,, \\ F_{it}(u) &:= \int_{\operatorname{Re} s = -1/4} \Gamma(-s)\,\Gamma(-2it - s)\,\Gamma(it + s + 1/2) u^s\, ds\,. \end{aligned}$$

We have to investigate the multivalued function $u \mapsto F_{it}(u)$ for $\arg u \in [-\pi, \pi]$.

Write $u = re^{i\psi}$, $r \geq 1$, $|\psi| \leq \pi$, and use $s = -\frac{1}{4} + iy$. We use Stirling's formula, and obtain the following estimate:

$$\begin{aligned} F_{it}(u) = &\int_{-\infty}^{\infty} \Gamma\left(\tfrac{1}{4} - iy\right) \Gamma\left(\tfrac{1}{4} - i(2t + y)\right) \Gamma\left(\tfrac{1}{4} + i(t + y)\right) \\ &\quad \cdot r^{-1/4 + iy} e^{(-1/4 + iy) i\psi}\, i\, dy \\ &\ll r^{-1/4} \int_0^\infty \left|\Gamma\left(\tfrac{1}{4} + i(t - y)\right) \Gamma\left(\tfrac{1}{4} - i(y + t)\right) \Gamma\left(\tfrac{1}{4} + iy\right)\right| e^{-\psi y + \psi t}\, dy \\ &\quad + r^{-1/4} \int_0^\infty \left|\Gamma\left(\tfrac{1}{4} + i(t + y)\right) \Gamma\left(\tfrac{1}{4} + i(y - t)\right) \Gamma\left(\tfrac{1}{4} - iy\right)\right| e^{\psi y + \psi t}\, dy \\ &\ll r^{-1/4} e^{\psi t} \int_0^\infty e^{-\frac{\pi}{2}(|t-y| + 2y + t) + |\psi| y} \big(1 + |t - y|\big)^{-1/4} (1 + t)^{-1/4} \\ &\qquad \cdot (1 + y)^{-1/4}\, dy \\ &\ll r^{-1/4}(1 + t)^{-1/2} e^{\psi t} \left(\int_0^t e^{(|\psi| - \pi/2) y - \pi t}\, dy + \int_t^\infty e^{(|\psi| - 3\pi/2) y}\, dy \right) \\ &\ll r^{-1/4}(1 + t)^{-1/2} e^{\psi t} \cdot \begin{cases} e^{-\pi t} t e^{(|\psi| - 3\pi/2) t} + e^{(|\psi| - 3\pi/2) t} & \text{if } |\psi| \geq \frac{\pi}{2} \\ e^{-\pi t} t + e^{(|\psi| - 3\pi/2) t} & \text{if } |\psi| \leq \frac{\pi}{2} \end{cases} \\ &\ll r^{-1/4}\,(1 + t)^{1/2}\, e^{\psi t} \cdot \begin{cases} e^{(|\psi| - 3\pi/2) t} & \text{if } |\psi| \geq \frac{\pi}{2} \\ e^{-\pi t} & \text{if } |\psi| \leq \frac{\pi}{2}\,. \end{cases} \end{aligned}$$

For $\vartheta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we obtain

$$\begin{aligned}
&H^{(2)}_{it}(re^{i\vartheta})H^{(2)}_{it}(re^{-i\vartheta})\\
&\qquad = \pi^{-5}e^{-2\pi t-2ir\cos\vartheta}\cosh^2(\pi t)(2r)^{2it}F_{it}(2re^{i(\pi/2+\vartheta)})F_{it}(2re^{i(\pi/2-\vartheta)})\\
&\qquad \ll r^{-1/2}\,(1+t)\,e^{\pi t}e^{(\pi/2+|\vartheta|-3\pi/2)t}e^{-\pi t} \ll r^{-1/2}\,(1+t)\,e^{(|\vartheta|-\pi)t}\,,\\
&H^{(1)}_{it}(re^{i\vartheta})H^{(1)}_{it}(re^{-i\vartheta})\\
&\qquad = \pi^{-5}e^{2\pi t+2ir\cos\vartheta}\cosh^2(\pi t)(2r)^{2it}F_{it}(2re^{i(\vartheta-\pi/2)})F_{it}(2re^{-i(\pi/2+\vartheta)})\\
&\qquad \ll e^{4\pi t}r^{-1/2}\,(1+t)\,e^{-\pi t}e^{(\pi/2+|\vartheta|-3\pi/2)t}e^{-\pi t} \ll r^{-1/2}\,(1+t)\,e^{(\pi+|\vartheta|)t}\,.
\end{aligned}$$

This yields for $y = re^{i\psi}$, $r \geq 1$, $-\pi \leq \psi \leq 0$:

$$\begin{aligned}
&P_{-it}(y) - P_{it}(y)\\
&\qquad \ll e^{\pi t}\big(e^{\pi t}H^{(2)}_{it}(re^{i(\psi+\pi/2)})H^{(2)}_{it}(re^{(-i(\psi+\pi/2))})\\
&\qquad\qquad - e^{-\pi t}H^{(1)}_{it}(re^{i(\pi/2+\psi)})H^{(1)}_{it}(re^{-i(\pi/2+\psi)})\big)\\
&\qquad \ll e^{\pi t}r^{-1/2}(1+t)e^{|\psi+\pi/2|t}\,.
\end{aligned}$$

□

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