Kloosterman sums for the modular group

Roelof W. Bruggeman

On the occasion of the sixtieth birthday of A. Schinzel

Abstract. This paper reviews results for Kloosterman sums $S_r(n,m;c)$ for the full modular group with the multiplier system v_r corresponding to real powers η^{2r} of the eta function of Dedekind. The emphasis is on the dependence on the parameter r, the weight.

Discussed are the asymptotic behavior of the sums $\sum_{c\leq T} S_r(n,m;c)/c$ and the continuation of the Kloosterman-Selberg series $L_r(n,m;s) = \sum_{c\geq 1} c^{-2s} S_r(n,m;c)$. The derivatives $\partial_r^k L_r(n,m;s)$ are continued to $(0,12)\times\mathbb{C}$; this continuation is meromorphic in s, and smooth in (r,s) outside the singularities. For integral values of the weight r, an explicit bound of $S_r(n,m;c)$ is given, that seems to be new.

1991 Mathematics Subject Classification: 11L05, 11L07, 11F37, 11F72, 11F20.

1. Kloosterman sums of real weight

This paper is an expanded version of my lecture at the International Number Theory Conference at Zakopane-Kościelisko, Poland, June 30 – July 9, 1997. Sections 1–3 contain the topics mentioned in the lecture. The later sections contain more details and proofs of new results.

I thank P. Kisbye for remarks on a previous version of this paper.

1.1. Definition

The Kloosterman sums discussed in this paper have the form

$$S_r(n,m;c) := \sum_{d \bmod c}^{*} e^{-\pi i r/2} v_r \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} e^{2\pi i(na+md)/c}, \tag{1.1}$$

for $c\in\mathbb{Z}_{\geq 1}$, $n,m,r\in\mathbb{R}$, $n\equiv \frac{r}{12} \bmod 1$. The sum is over representatives of $\mathbb{Z}/c\mathbb{Z}$ relatively prime to c, and v_r is the multiplier system of the $2r$-th power of the eta

For $r = 0$, and $n, m \neq 0$, we obtain the classical Kloosterman sum, used as a tool in Kloosterman's study of quadratic forms in [14]. Actually, Kloosterman used a more general sum, in which d is restricted by a congruence condition. Shortly afterwards, Kloosterman worked with Hecke, and the role of modular cusp forms became more explicit, see [15]. Kloosterman used the circle method to relate Fourier coefficients of cusp forms to Kloosterman sums. Later, Petersson gave series expansions for the Fourier coefficients of Poincaré series of the form $\sum_{c=1}^{\infty} c^{-1} S_0(n,m;c) j(n,m,c)$, where j can be expressed in a Bessel function ([23], equation (g) on p. 178). So the size of $S_0(n,m;c)$ is important to estimate Fourier coefficients of automorphic forms.

This relation between Kloosterman sums and automorphic forms allows wide generalizations. Here I restrict myself to those Kloosterman sums that belong to the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

In general, automorphic forms on the upper half-plane for a discrete subgroup Δ of $\mathrm{SL}_2(\mathbb{R})$ satisfy the following relation for all $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Delta$:

$$f\left(\frac{az+b}{cz+d}\right) = v\begin{pmatrix} a & b \\ c & d\end{pmatrix}(cz+d)^k f(z). \tag{1.2}$$

To admit non-zero solutions, the map $v : \Delta \to \mathbb{C}^*$ has to be a *multiplier system*. (For non-integral weight, a multiplier system is not a character of Δ, as it has to compensate for the choice of the argument of $cz+d$.) For the modular group Γ there is one family $\mathbb{C}/12\mathbb{Z} : r \mapsto v_r$ of multiplier systems; v_r is suitable for weights $k \equiv r \bmod 2$.

These multiplier systems are explicitly known. One finds the essential results already in Dedekind's note [8]; see also [26]. This leads to the following description of the multiplier system v_r, in terms of the *Dedekind sum* $S(d,c)$:

$$\begin{aligned} v_r\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix} &= e^{\pi i r n/6} \\ v_r\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= e^{\pi i r(a+d)/6c \mp \pi i r/2 - 2\pi i r S(d,c)} \qquad \text{if } \pm c > 0, \\ S(d,c) &= \sum_{x \bmod c} \left(\left(\frac{x}{c}\right)\right)\cdot\left(\left(\frac{dx}{c}\right)\right) \qquad\qquad (1.3) \\ ((x+n)) &= \begin{cases} 0 & \text{if } x = 0, n \in \mathbb{Z}, \\ x - \frac{1}{2} & \text{if } 0 < x < 1,\ n \in \mathbb{Z}. \end{cases} \end{aligned}$$

A more general definition of Kloosterman sum for a discrete group Γ (having ∞ as a cusp with width 1) and a multiplier system v is $\sum_\gamma v(\gamma)^{-1} e^{2\pi i(na+md)/c}$ where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$ runs through representatives of $\Gamma_\infty\backslash\Gamma(c)/\Gamma_\infty$, with $\Gamma_\infty = \{\gamma \in \Gamma : \gamma\cdot\infty = \infty\}$ and $\Gamma(c) = \left\{\left(\begin{smallmatrix} * & * \\ c & *\end{smallmatrix}\right) \in \Gamma\right\}$, $c \geq 1$. For the modular case, we take out the common factor $e^{-\pi i r/2}$, and arrive at the following variant of (1.1):

$$S_r(n,m;c) = \sum\nolimits^*_d e^{2\pi i r S(d,c) + 2\pi i((n-r/12)a + (m-r/12)d)/c}. \tag{1.4}$$

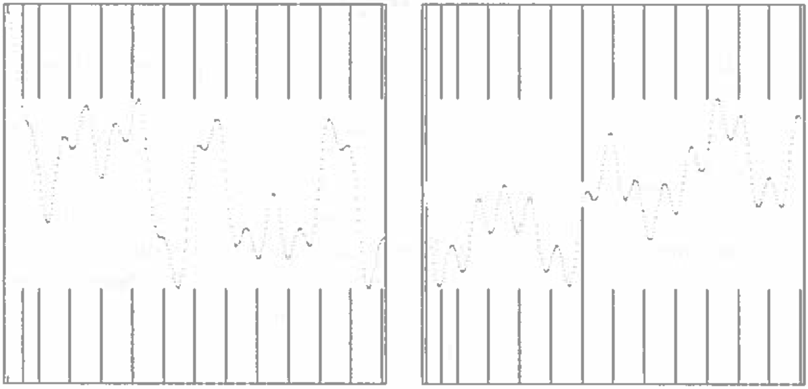

Figure 1: Graphs of $r \mapsto S_r\left(\frac{r}{12}, \frac{r}{12}; 17\right)$ (left) and $r \mapsto S_r\left(\frac{r}{12}, \frac{r}{12}+1; 17\right)$ (right) on $[-.176, 12.176]$. Vertical axis: $[-16.320, 16.320]$.

These Kloosterman sums satisfy the following relations:

$$S_r(n,m;c) = S_r(n+c,m;c) = S_r(n,m+c;c) \tag{1.5}$$
$$= S_r(m,n;c) \tag{1.6}$$
$$= S_{-r}(-n,-m;c) = \overline{S_r(n,m;c)}, \tag{1.7}$$
$$S_r\left(\nu+\frac{r}{12}, \mu+\frac{r}{12}; c\right) = S_{r+12}\left(\nu+\frac{r}{12}, \mu+\frac{r}{12}; c\right) \tag{1.8}$$
$$= S_{r+12c}(\nu+c+\frac{r}{12}, \mu+c+\frac{r}{12}; c). \tag{1.9}$$

Relation (1.5) is clear from (1.4). For (1.6), use that the variable substitution $x \mapsto ax$ in (1.3) yields $S(a,c) = S(d,c)$. The substitution $x \mapsto -x$ leads to (1.7). The multiplier system depends on r mod $12\mathbb{Z}$; this gives (1.8). Relation (1.9) follows from the fact that $12cS(d,c) \in \mathbb{Z}$ (see, e.g., Theorem 2 on p. 27 of [26]).

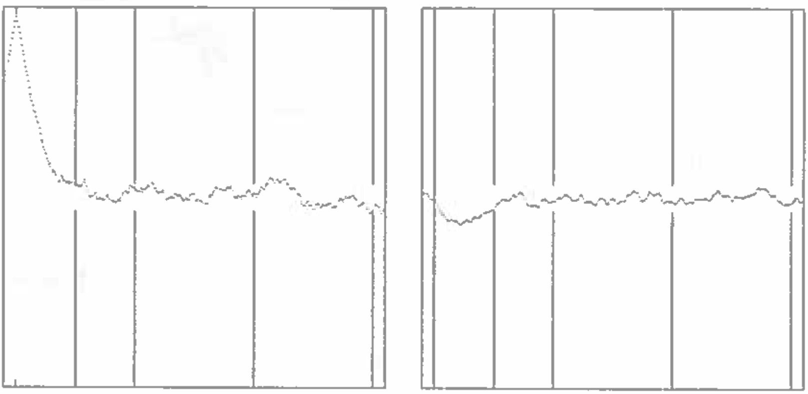

Figure 2: Graphs of $r \mapsto \left(\frac{r}{12}, \frac{r}{12}; 761\right)$ (left) and $r \mapsto S_r\left(\frac{r}{12}, \frac{r}{12}+1; 761\right)$ (right)

1.2. Graphs

Let $\kappa \in \mathbb{Z}$. The function $r \mapsto S_r\left(\frac{r}{12}, \frac{r}{12} + \kappa; c\right)$ is a Fourier polynomial, with trivial bound

$$\left|S_r\left(\frac{r}{12}, \frac{r}{12} + \kappa; c\right)\right| \leq S_0(0,0;c) = \varphi(c), \tag{1.10}$$

the number of elements of $(\mathbb{Z}/c\mathbb{Z})^*$.

Figure 1 gives the graphs for $c = 17$ and κ equal to 0 and 1. The maximal value 16 is attained for $\kappa = r = 0$. Except in a neighborhood of 0, for $\kappa = 0$, the absolute value is much smaller than given by (1.10). For larger values of c the denominators in the exponent are larger, so the oscillation is more rapid. In Figure 2, one finds Kloosterman sums for $c = 761$, on a shorter r-interval.

1.3. Bounds

The vertical lines in Figures 1 and 2 correspond to known estimates. They indicate windows through which the graph has to pass. I have indicated only windows for which I know a reference or a proof: at integral values of r, and at $r = \pm\frac{1}{2}$. In the literature, I have not found similar estimates at other half-integral values of r. Possibly, methods and results from [18] and [22] can be used.

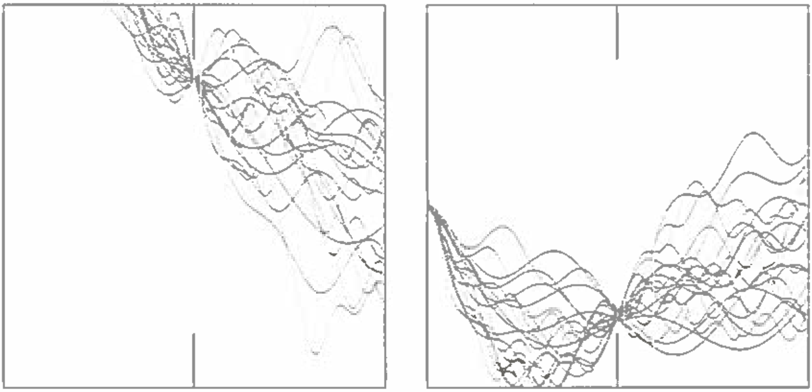

Figure 3: Graphs of $S_r(r/12, r/12; p)/2\sqrt{p}$ (left) and $S_r(r/12, 1+r/12; p)/2\sqrt{p}$ (right) for $0 \leq r \leq 1$ and primes $p \in [20, 120]$. The lines indicate the window $[-1, 1]$ on the vertical scale at $r = \frac{1}{2}$.

A review of these estimates starts with the classical case $r = 0$. Kloosterman, [14], showed $S_0(n, m; c) = O(c^{3/4+\varepsilon})$. A better estimate is

$$|S_0(n, m; c)| \leq \sigma_0(c)(n, m, c)^{1/2} c^{1/2}, \tag{1.11}$$

where $\sigma_0(c)$ is the number of divisors of c. Salié, [27], reduced the problem to the case $c = p^k$ with p prime, by a multiplicativity relation, and showed that if cases that p divides n or m are easier. By methods from algebraic geometry, Weil treated the case $c = p$ in [32]. Estermann, [10], combined the results and gave the formulation (1.11).

For integral values of r, the estimate $S_r(n,m;c) = O(c^{1/2+\varepsilon})$ is due to Myerson, [20]. The technique is to reduce these Kloosterman sums to classical ones ($r = 0$) with c replaced by $c' \in \{c, 3c, 4c, 12c\}$. This O-statement is sufficient for the purpose of [20]. In Subsection 4.1, I show how to use the methods of Salié and Estermann to obtain the explicit bounds used in the graphs.

In the study of the partition functions, Rademacher discussed in Chapter 14 of [25] the Kloosterman sums $S_{-1/2}\left(-\frac{1}{24}, -\frac{1}{24}+n;k\right) = S_{1/2}\left(\frac{1}{24}, \frac{1}{24}-n;k\right)$. (The reciprocal $\frac{1}{\eta}$ of the eta function of Dedekind is essentially the generating function for the number of partitions.) In §122–126 of [25], Rademacher quotes work of Whiteman, mentions a formula of Selberg, and gives explicit formulas that imply an estimate of the type $S_{\pm 1/2}(n,m;c) = O_{n,m}(c^{1/2+\varepsilon})$. I have indicated the resulting bounds in the figures. Subsection 4.2 gives a more detailed discussion.

Vardi, [31], §5, conjectured that $S_r(n,m;c) = O(c^{1-1/k+\varepsilon})$ for rational $r \notin \mathbb{Z}$, with k the denominator of r.

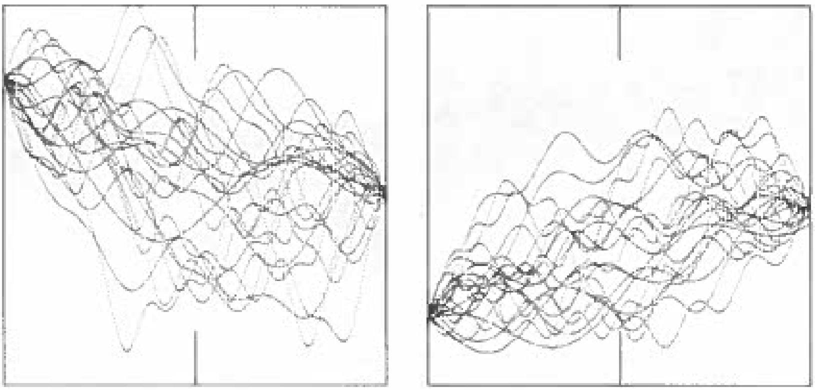

Figure 4: Graphs of $S_r(r/12, r/12; p)/2\sqrt{p}$ (left) and $S_r(r/12, 1+r/12; p)/2\sqrt{p}$ (right) for $\frac{1}{2} \leq r \leq \frac{3}{2}$ and primes $p \in [20, 120]$. The lines indicate the window $[-1, 1]$ on the vertical scale at $r = 1$.

To see whether $r = \frac{1}{2}$ is special, Figure 3 gives scaled graphs of $r \mapsto S_r(\frac{r}{12}, \frac{r}{12}; p)$ and $r \mapsto S_r(\frac{r}{12}, 1+\frac{r}{12}; p)$ for the primes $p \in [20, 120]$. In both graphs, we see that for r near $\frac{1}{2}$ the Kloosterman sums leave the window prescribed for $r = \frac{1}{2}$. In the case of $S_r(\frac{r}{12}, \frac{r}{12}; p)$, the Kloosterman sums come down from the value $p-1$ at $r = 0$. In the other case, the value -1 at $r = 0$ is recognizable. At $r = \frac{1}{2}$, the graphs seem to go through approximately the same point. This is in accordance with Rademacher's value (§126 of [25]):

$$S_{1/2}\left(\frac{1}{24}, \kappa + \frac{1}{24}; p\right) = 2\sqrt{p}\left(\frac{3}{-}\right)\cos\frac{4\pi w}{} \qquad (1.12)$$

if $\left(\frac{1+24\kappa}{p}\right)=1$. The number w satisfies $(24w)^2\equiv 1+24\kappa \bmod p$. If we work this out, we find $2\sqrt{p}\cos(\frac{\pi}{6}\pm\frac{\pi}{6p})$ for $\kappa=0$ and $2\sqrt{p}\cos(\frac{5\pi}{6}\pm\frac{5\pi}{6p})$ for $\kappa=1$. Indeed, the graphs go approximately through the level $\pm\frac{1}{2}\sqrt{3}$.

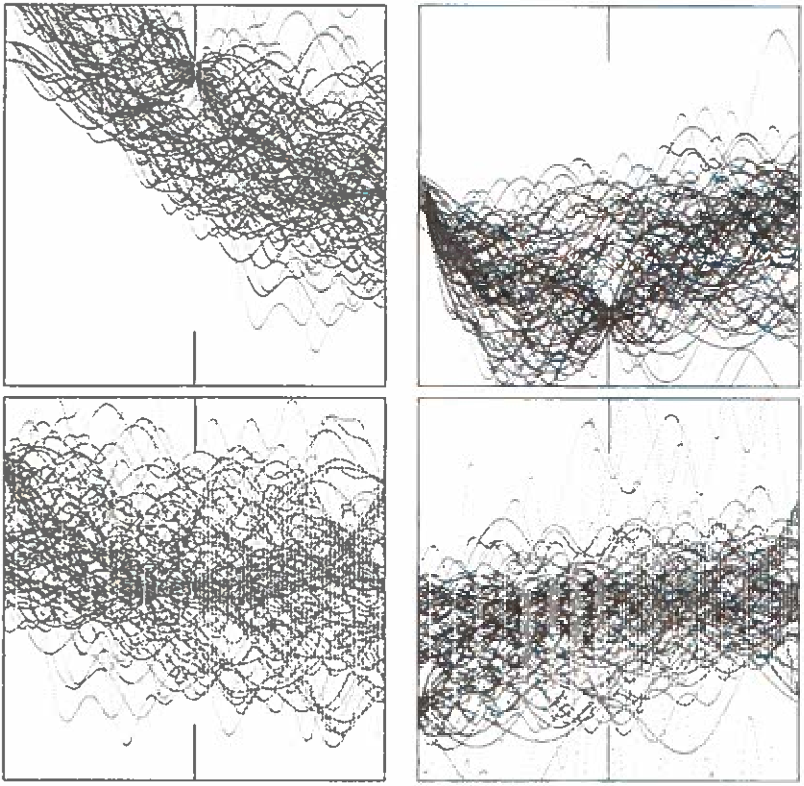

Figure 5: Graphs of $S_r(r/12,\kappa+r/12;c)/f_r(\kappa;c)$, with $\kappa=0$ (left) and $\kappa=1$ (right), for $r\in[0,1]$ (top) and $r\in\left[\frac{1}{2},\frac{3}{2}\right]$ (bottom). The bound $\left|S_r\left(\frac{r}{12},\frac{r}{12}+\kappa;c\right)\right|\le f_r(\kappa;c)$ at $r=1$ is given in Proposition 4.1. At $r=\frac{1}{2}$, the bound is taken from [25], §125 (multiplicativity) and §126 (prime powers), with $\cos(4\pi r/p^\lambda)$ estimated by 1.

Figure 4 gives similar information on the r-interval $\left[\frac{1}{2},\frac{3}{2}\right]$. The bound $2\sqrt{p}$ can be found in Proposition 4.1. At $r=1$, the Kloosterman sums fill up a larger part of the window that at $r=\frac{1}{2}$; outside $r=1$, some of the Kloosterman sums leave the window. In Figure 4, the values at $r=\frac{1}{2}$ are visible. There seem to be special values at $r=\frac{3}{2}$ as well. In the literature, I have not found a formula like that of Rademacher for other $r\in\frac{1}{2}+\mathbb{Z}$ than $\pm\frac{1}{2}$.

Figure 5 gives the result of the same computations for all $c\in[20,120]$. We see that the window prescribed by the bounds is filled more completely.

I do not think that these computations give any indication concerning the truth of the conjecture of Vardi mentioned above.

2. Kloosterman-Selberg series

The *Kloosterman-Selberg series*

$$L(n,m;r,s) := \sum_{c=1}^{\infty} S_r(n,m;c)c^{-2s}, \tag{2.1}$$

with $r \in \mathbb{R}$, $n \equiv m \equiv \frac{r}{12} \bmod 1$, converges absolutely for $\operatorname{Re} s > 1$. Selberg, [29], defined it and proved its meromorphic continuation to $s \in \mathbb{C}$. The most interesting case is $nm \neq 0$. For that case, Goldfeld and Sarnak, [11], gave the estimate $L(n,m;r,s) = O_\sigma(|mn||s|^{1/2})$ on vertical lines $\operatorname{Re} s = \sigma > \frac{1}{2}$, outside a neighborhood of the poles (which occur on $(\frac{1}{2},1]$ if they occur at all in $\operatorname{Re} s > \frac{1}{2}$). Yoshida, [34], gave the improvement $O(|mn|^{1/2}|s|^{1/2})$, for the case $r = 0$.

The singularities are related to the spectrum of the Casimir operator in the space of functions on the upper half-plane that transform according to the multiplier system v_r, and are square integrable on a fundamental domain. For the modular group, the exceptional eigenvalues in this spectrum are known. This knowledge implies that the only singularities of $L(n,m;r,s)$ (with $(n,m) \neq (0,0)$, $0 \leq r < 12$) in the half-plane $\operatorname{Re} s > \frac{1}{2}$ are first order ones in the following cases:

condition	position	residue
$0 < r < 1$	$1 - \frac{r}{2}$	$\dfrac{p_{n-r/12}(r)p_{m-r/12}(r)}{\pi 2^r N_\eta(r)}$
$11 < r < 12$	$\frac{r}{2} - 5$	$\dfrac{p_{r/12-n-1}(12-r)p_{r/12-m-1}(12-r)}{\pi 2^{12-r} N_\eta(12-r)}$

(2.2)

(In view of (1.8), results for $0 \leq r < 12$ cover all cases.) The quantities p_ν and N_η are related to the eta function of Dedekind on the upper half-plane $\mathfrak{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$ in the following way:

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right), \tag{2.3}$$

$$\eta(z)^{2r} = e^{\pi i r z/6} \sum_{\nu=0}^{\infty} p_\nu(r) e^{2\pi i \nu z}, \tag{2.4}$$

$$p_\nu(r) := 0 \qquad \text{if } \nu < 0, \tag{2.5}$$

$$N_\eta(r) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} y^r \, |\eta(z)|^{4r} \, \frac{dx\,dy}{y^2}. \tag{2.6}$$

On the domain of absolute convergence $\operatorname{Re} s > 1$, the quantity $L(n,m;r,s)$ is continuous in $r \in \mathbb{R}$, provided we let n and m vary with r such that $n \equiv m \equiv \frac{r}{12} \bmod 1$ stays valid. The estimate $S(d,c) = O(c)$ implies that we have k times

$$L_k(n,m;r,s) := \sum_{c=1}^{\infty} c^{-2s} \sum_{d \bmod c}^{*} S(d,c)^k e^{2\pi i r S(d,c) + 2\pi i(\nu a+\mu d)/c} \tag{2.7}$$
$$= (2\pi i)^{-k} \partial_r^k L(n,m;r,s). \tag{2.8}$$

Proposition 2.1. *Let $r \in \mathbb{R}$, $k \in \mathbb{Z}$, $k \geq 0$, $n \equiv m \equiv \frac{r}{12} \bmod 1$. The function $s \mapsto L_k(n,m;r,s)$ has a meromorphic extension to $s \in \mathbb{C}$.*

i) *Case $r = 0$.*

a) *$L_0(0,0;0,s) = \zeta(2s-1)/\zeta(2s)$ is holomorphic on $\mathrm{Re}\, s > \frac{1}{2}$, except for a first order pole at $s = 1$; if $nm \neq 0$, then $L_0(n,m;0,s)$ is holomorphic on $\mathrm{Re}\, s > \frac{1}{2}$.*

b) *If $k \geq 2$ is even, then singularities in $\mathrm{Re}\, s > \frac{1}{2}$ can occur only at half-integral points in the interval $\left[1, \frac{k+1}{2}\right]$. At $\frac{k+1}{2}$ there is a first order pole with residue $\left(\frac{1}{12}\right)^k \sigma_{-k}(n)\sigma_{-k}(m)\zeta(k+1)^{-2}$. (Here ζ denotes the zeta function of Riemann, and $\sigma_a(N) = \sum_{d|N,\, d>0} d^a$ if $N \neq 0$, and $\sigma_a(0) = \zeta(-a)$.)*

c) *If $k \geq 1$ is odd, then singularities in $\mathrm{Re}\, s > \frac{1}{2}$ can occur only at half-integral points in $\left[1, \frac{k}{2}\right]$.*

ii) *Case $0 < r < 12$.*

Singularities of $L_k(n,m;r,s)$ in $\mathrm{Re}\, s > \frac{1}{2}$ can occur only at $1 - \frac{r}{2}$ if $0 < r < 1$, and at $\frac{r}{2} - 5$ if $11 < r < 12$.

If $0 < r < 1$, then

$$\lim_{s \to 1-r/2} \left(s + \frac{r}{2} - 1\right)^{k+1} L_k(n,m;r,s)$$
$$= \frac{k!}{\pi} p_{n-r/12}(r) p_{m-r/12}(r) 2^{-r-k} (-2\pi i)^{-k} N_\eta(r)^{-1},$$

and if $11 < r < 12$, then

$$\lim_{s \to r/2-5} \left(s + 5 - \frac{r}{2}\right)^{k+1} L_k(n,m;r,s)$$
$$= \frac{k!}{\pi} p_{r/12-n-1}(r) p_{r/12-m-1}(r) 2^{r-12-k} (2\pi i)^{-k} N_\eta(12-r)^{-1}.$$

According to (1.8), the results for $0 \leq r < 12$ cover all cases. See (2.4)–(2.6) for p_ν and N_η.

The case $r = k = 0$ is classical, see Selberg [29]. The results for $k \geq 1$, $r = 0$, can be found in [2], Proposition 4.2, and [5], Proposition 7.1. In Subsection 5.4, I shall prove the statements in Part ii). Actually, the proof for the case $0 < r < 12$ is easier than for Part i), b) and c). But Subsection 5.4 gives an impression of the method that works for $r = 0$ as well.

Consider an even integer $k \geq 2$. The Dirichlet series for $L_k(0,0;0,s)$ has positive coefficients $\sum_{d \bmod c}^{*} S(d,c)^k$. This shows that $L_k(n,m;r,s)$ has $\mathrm{Re}\, s > \frac{k+1}{2}$ as domain of absolute convergence. If $r \in (0,12)$, there is a wide strip, containing $\frac{1}{2} < \mathrm{Re}\, s \leq \frac{k+1}{2}$, on which the function $L_k(n,m;r,s)$ is holomorphic, but the series not absolutely convergent. It might be interesting to obtain estimates on

3. Sums of Kloosterman sums

Singularities of the Kloosterman-Selberg series are related to eigenvalues of the Casimir operator $C_r = -y^2\partial_x^2 - y^2\partial_y^2 + iry\partial_x$ (with $z = x + iy \in \mathfrak{H}$) in the space of square integrable functions satisfying (1.2). Eigenvalues of C_r between 0 and $\frac{1}{4}$ are called exceptional. Selberg, [29], discussed the question whether exceptional eigenvalues of C_0 are absent for congruence subgroups. For $\Gamma_0(N)$, this would follow if

$$\sum_{c\leq T,\, c\equiv 0 \bmod N} \frac{1}{c} S_0(n,\mu;c) = O(T^\varepsilon) \qquad (T\to\infty) \tag{3.1}$$

would hold for each $\varepsilon > 0$, and all integral n and m such that $nm \neq 0$. This is equivalent to $\sum_{c\leq T,\, c\equiv 0 \bmod N} S_0(n,m;c) = O(T^{1+\varepsilon})$. This had been conjectured by Linnik, [17], in the case $N = 1$.

In the present situation, let us consider

$$K_r(n,m;T) := \sum_{1\leq c\leq T} \frac{1}{c} S_r(n,m;c), \tag{3.2}$$

with $r, n, m, T \in \mathbb{R}$, $n \equiv m \equiv \frac{r}{12} \bmod 1$, T large, and c running through positive integers. A trivial estimate is $K_r(n,m;T) = O(T)$. In this section, I mention some non-trivial estimates of this quantity. (Derivatives of $K_r(n,m;T)$ with respect to r are interesting as well, but I have got trivial estimates only.)

There are two, related, tools to obtain non-trivial estimates of $K_r(n,m;T)$. The first tool is the sum formula of Kuznetsov, see [16]. Kuznetsov's result is

$$K_0(n,m;T) = O_{n,m}\left(T^{1/6}(\log T)^{1/3}\right) \tag{3.3}$$

for $nm \neq 0$. Deshouillers and Iwaniec, [9], used the sum formula of Kuznetsov in weight 0, and sum also over n and m. They worked in the more general situation of congruence subgroups.

Goldfeld and Sarnak, [11], used the Kloosterman-Selberg series. For $0 < r < 12$, their result leads to an estimate of the form

$$K_r(n,m;T) = E_r(n,m;T) + O_{n,m}\left(T^{\beta/3+\varepsilon}\right) \tag{3.4}$$

$$E_r(n,m;T) = \begin{cases} \dfrac{p_\nu(r)p_\mu(r)}{\pi(1-r)2^{r-1}N_\eta(r)} T^{1-r} & \text{if } 0 < r < 1, \\ 0 & \text{if } 1 \leq r \leq 11, \\ \dfrac{p_{-\nu-1}(12-r)p_{-\mu-1}(11-r)}{\pi(r-11)2^{12-r}N_\eta(12-r)} T^{r-11} & \text{if } 11 < r < 12, \end{cases} \tag{3.5}$$

provided we have an estimate $|S_r(n,m;c)| = O(c^\gamma)$ as $c \to \infty$ for each $\gamma > \beta$. Moreover, they state that the dependence on n and m can be shown to be given by a factor $|nm|$. Yoshida's improvement, [34], would give a factor $|nm|^{1/2}$. The assumption on the individual Kloosterman sums means that this result does not

In pictures, I prefer to work with

$$H_r(n,m;T) := K_r(n,m;T) - K_r(n,m;T/2) = \sum_{T/2<c\le T} \frac{1}{c} S_r(n,m;T). \tag{3.6}$$

Asymptotic estimates for K_r and for H_r are easily transformed into each other. If the term corresponding to $E_r(n,m;T)$, for $0 < r < 1$, dominates the error term, we expect that $H_r(n,m;T)$ behaves exponentially in T.

Figure 6 shows a definite difference between the regions $r \in (0,1)$ and $r \in (1,4]$. On $(0,1)$ the quantity $H_r\left(\frac{r}{12}, \frac{r}{12}; 50 \cdot 2^j\right)$ increases more or less exponentially with j, whereas its behavior for larger r does not seem to have much structure.

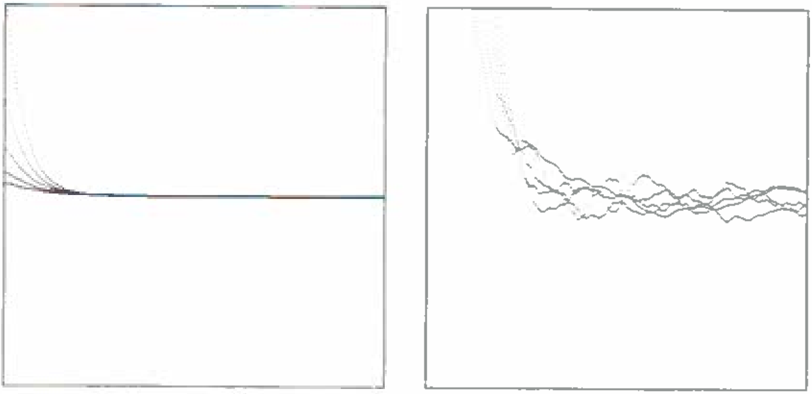

Figure 6: Superposition the graphs of $H_r\left(\frac{r}{12}, \frac{r}{12}; T\right)$ for $T = 50$, 100, 200, 400, and 800, as a function of $r \in [0,4]$. Vertical scale $[-250, 250]$ (left), and $[-5, 5]$ (right).

For $\tau \in C_c^\infty(0,\infty)$, we consider the quantity

$$H_r^{\mathrm{sm}}(n,m;\tau,T) := \sum_{c=1}^{\infty} \frac{1}{c} S_r(n,m;c)\tau(T/c). \tag{3.7}$$

If we choose τ as an approximation of the characteristic function of the interval $[1,2)$, this quantity is an approximation of $H_r(n,m;T)$. The sum formula of Kuznetsov gives, for $0 < r < 12$, $n = \nu + \frac{r}{12}$, $m = \mu + \frac{r}{12}$, and $T > 4\pi\sqrt{|nm|}$:

$$\begin{aligned} &H_r^{\mathrm{sm}}(n,m;\tau,T) &(3.8)\\ &= E_r^{\mathrm{sm}}(n,m;\tau,T) + O_\tau\left(\sqrt{1+|n|}\sqrt{1+|m|}\left(1+\log\frac{T}{4\pi\sqrt{|nm|}}\right)\right), \end{aligned}$$

$$\begin{aligned} &E_r^{\mathrm{sm}}(n,m;\tau,T) &(3.9)\\ &= \begin{cases} \dfrac{2^{1-r}\mathcal{M}\tau(r-1)p_\nu(r)p_\mu(r)}{\pi N_\eta(r)} T^{1-r} & \text{if } 0<r<1, \\ 0 & \text{if } 1\le r\le 11, \\ \dfrac{2^{r-11}\mathcal{M}\tau(11-r)p_{-\nu-1}(12-r)p_{-\mu-1}(12-r)}{\cdots} T^{r-11} & \text{if } 11<r<12. \end{cases} \end{aligned}$$

$$\mathcal{M}\tau(u) := \int_0^\infty y^u \tau(y) \frac{dy}{y}. \tag{3.10}$$

(This is an immediate application of Lemma 4.2 in [3]: take r equal to $12n$, $X = 4\pi\sqrt{|nm|}/T$, and $\kappa = \mu - \nu$.)

These sums of Kloosterman sums with smooth bounds have an asymptotic behavior with an error term that is even better than the conjecture of Linnik prescribes. This seems to be the general situation; Miatello and I found it also in the context of Lie groups of real rank one, see [6].

In the transition from a sum with smooth bounds to a sum with sharp bounds, the steepness of τ has to be taken into account. The resulting changes in the error term, and in the term E, are not difficult to handle. The two short sums in the difference $H_r^{\rm sm}(n,m;\tau,T) - H_r(n,m;T)$ are crucial. Up till now, we cannot get our hands on the cancellation between Kloosterman sums in these short sums. To get a non-trivial error term at all, we need (1) a non-trivial bound of individual Kloosterman sums and (2) some information on the spacing of the numbers c occurring in the sum. Then we adjust the steepness of τ to the size of T. This has a considerable effect on the error term. In the context of this paper, point (2) is trivial. But a non-trivial bound is available only for special values of r.

Figure 6 and similar numerical results leave me with the impression that Linnik's size $O(T^\varepsilon)$ of the error term is the right one, and that we should look for better methods to handle the transition from smooth to sharp bounds.

4. Bounds for Kloosterman sums with special values of the weight

4.1. Integral weight

Estermann, [10], has streamlined the results of Salié, [27], and Weil, [32], concerning the Kloosterman sums $S_0(n,m;c)$. Myerson, [20], shows how the estimation of $S_r(n,m;c)$ with $r \in \mathbb{Z}$ can be reduced to these results of Salié, Weil and Estermann. The next proposition gives, for integral r, the explicit bounds used in the pictures in Subsection 1.3.

Proposition 4.1. *Let $r, c \in \mathbb{Z}$, $n \equiv m \equiv \frac{r}{12} \bmod 1$, and $c \geq 1$. Define the numbers $f_p(\nu,\mu;e) \geq 0$ as indicated in Table 1. Then*

$$|S_r(n,m;c)| \leq \prod_p f_p(v_p(12n), v_p(12m); v_p(c)),$$

where v_p is the valuation associated to the prime p.

Proof. See Lemmas 4.3–4.7. Note that $\gamma \mapsto v_r(\gamma)$ is a character of $\mathrm{SL}_2(\mathbb{Z})$ if $r \in \mathbb{Z}$.

e	$\nu=\mu=0$	$\nu+\mu>0,\ \nu\mu=0$	$\nu\ge1,\ \mu\ge1$
0	1		
1	$p-1$ if $p\le5$	1 if $p\ge5$	$p-1$
1	$2\sqrt{p}$ if $p\ge7$	0 if $p\le3$	
≥2	$2p^{e/2}$ if $p\ge5$	0	$p\,f_p(\nu-1,\mu-1;e-1)$
≥2	$2\cdot3^{(e-1)/2}$ if $p=3$		
2	2 if $p=2$		
≥3, odd	$2\cdot2^{(e+1)/2}$ if $p=2$		
≥4, even	$2\cdot2^{e/2}$ if $p=2$		

Table 1: The local bounds $f_p(\nu,\mu;e)$ in Proposition 4.1.
The definition is partly recursive.

Corollary 4.2. *Let* $\mathrm{pr}(c)$ *denote the number of different primes dividing* c. *For* n, m, r *and* c *as in Proposition* 4.1, *we have*

$$S_r(n,m;c)=O\big(2^{\mathrm{pr}(c)}(12n,12m,c)^{1/2}c^{1/2}\big).$$

Proof. Inspection of Table 1. □

Lemma 4.3. *Let* $n\equiv m\equiv\frac{r}{12}\bmod 1$. *Then*

$$S_r(n,m;c)=\varphi(c)\varphi(12c)^{-1}e^{-\pi ir/2}\sum_{d\bmod 12c}^{*}v_r\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}e^{2\pi i(na+md)/c},\qquad(4.1)$$

where for each (d,c) *the matrix* $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ *is chosen in* $\Gamma^0(12)$.

As usual, $\Gamma^0(M)=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z}):b\equiv0\bmod M\}$, and $\varphi(k)=|(\mathbb{Z}/k\mathbb{Z})^*|$ is Euler's function.

Proof. In the sum in (1.1), we let d run over representatives of $(\mathbb{Z}/12c\mathbb{Z})^*$ instead of $(\mathbb{Z}/c\mathbb{Z})^*$. The factor $\varphi(c)/\varphi(12c)$ corrects the repetitions. We use that $v_r\left(\begin{smallmatrix}a&b+a\\c&d+c\end{smallmatrix}\right)=v_r\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)e^{\pi ir/6}$, and that $m\equiv\frac{r}{12}\bmod 1$.

Each d is relatively prime to 12. So left multiplication by a suitable $\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)$ brings the matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ into $\Gamma^0(12)$. We use that $v_r\left(\begin{smallmatrix}a+c&b+d\\c&d\end{smallmatrix}\right)=v_r\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)e^{\pi ir/6}$. □

Lemma 4.4. *Define* $\chi_3[c,d]:=e^{2\pi icd/3}$ *and* $\chi_4[c,d]:=e^{3\pi i(cd-d+1)/2}$ *for* $c,d\in\mathbb{Z}$. *For each* $r\in\mathbb{Z}$, $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma^0(12)$ *we have*

$$v_r\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}=\chi_4[c,d]^r\cdot\chi_3[c,d]^r.$$

Proof. Some computations show that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi_k[c,d]$ determines a character of $\Gamma^0(k)$ for $k = 4$ and $k = 3$. (Use that $a \equiv d \bmod k$ and $d^2 \equiv 1 \bmod k$.) A further computation on the generators $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ shows that $\chi_4[c,d] = v_3\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $\Gamma^0(4)$ and $\chi_3[c,d] = v_8\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $\Gamma^0(3)$. Use $v_r^{-1} = v_{3r} \cdot v_{8r}$ and $\Gamma^0(12) = \Gamma^0(4) \cap \Gamma^0(3)$ to obtain the lemma. □

Lemma 4.5. *Define, for $r, N, M, u, k \in \mathbb{Z}$, $k \geq 0$:*

$$T_2(N,M;r,u;2^k) := \sum_{x \bmod 2^k}^{*} \chi_4[u,x]^r e^{2\pi i(N\bar{x}+Mx)/2^k},$$

$$T_3(N,M;r,u;3^k) := \sum_{x \bmod 3^k}^{*} \chi_3[u,x]^r e^{2\pi i(N\bar{x}+Mx)/3^k}$$

$$T_p(N,M;r,u;p^k) := S_0(N,M;p^k) \quad \textit{for primes } p \geq 5,$$

with $x\bar{x} \equiv 1 \bmod p^k$. If $n \equiv m \equiv \frac{r}{12} \bmod 1$ and $12c = \prod_p p^{d_p}$, then there are integers ε_p, ζ_p and u_p satisfying $(\varepsilon_p, p) = (\zeta_p, p) = 1$, such that

$$S_r(n,m;c) = (-i)^r \varphi(c)\varphi(12c)^{-1} \prod_p T_p(12n\varepsilon_p, 12m\zeta_p; r, u_p; p^{d_p}).$$

Proof. This is the usual multiplicativity of Kloosterman sums, see [14], 2.41–42. Lemma 4.4 gives the multiplicativity of v_r. □

d	p	$\nu=\mu=0$	$\nu+\mu>0,\ \nu\mu=0$	$\nu\geq 1,\ \mu\geq 1$
0	≥ 5		1	
1	≥ 5	$2p^{1/2}$	1	$p-1$
1	≤ 3		$p-1$	
≥ 2	≥ 3	$2p^{d/2}$	0	$p\,t_p(\nu-1,\mu-1;d-1)$
2	2		2	
3	2	4	0	$p\,t_p(\nu-1,\mu-1;d-1)$
≥ 4 (even)	2	$4p^{d/2}$	0	$p\,t_p(\nu-1,\mu-1;d-1)$
≥ 5 (odd)	2	$4p^{(d+1)/2}$	0	$p\,t_p(\nu-1,\mu-1;d-1)$

Table 2: Values of $t_p(\nu,\mu;d)$ used in Lemma 4.6.

Lemma 4.6. *Let t_p be chosen as indicated in Table 2, and let v_p denote the valuation at the prime p. Then $\left|T_p(N,M;r,u;p^d)\right| \leq t_p(v_p(N), v_p(M); d)$ for all $N, M \in \mathbb{Z}$, and all $d \geq 0$ satisfying $d \geq 1$ if $p = 3$, and $d \geq 2$ if $p = 2$.*

Proof. A trivial bound for $T_p(N,M;r,u;p^d)$ is $\varphi(p^d)$. If $d = 1$, and $v_p(N) = v_p(M) = 0$, Weil's estimate, $|S_0(N,M;p)| \leq 2\sqrt{p}$, see [32], gives an improvement

Now we consider $d \geq 2$. The reasoning is the same as that of Estermann, [10]. The basis decomposition is

$$\begin{aligned} &T_p(N, M; u, p^d) \\ &= \sum_{x \bmod p^e}^{*} \chi_k[u,x]^r e^{2\pi i(N\bar{x}+Mx)/p^d} \sum_{y \bmod p^{d-e}} e^{2\pi i(My+NY)/p^{d-e}}, \end{aligned} \tag{4.2}$$

with $1 \leq e < d$, and $e \geq 2$ if $p = 2$. We have denoted $Y = -\bar{x}^2 y + p^e \bar{x}^3 y^2 - p^{2e}\bar{x}^4 y^3 + \ldots$. This sum can be terminated as soon as the exponent of p reaches $d - e$.

In the case that p divides N as well as M, take $e = d - 1$, to obtain a recursion formula. If p divides exactly one of N and M, the same choice leads to the value 0.

Let $v_p(N) = v_p(M) = 0$. If $p = 2$ and $d = 2$ or 3, we use the trivial estimate.

For $d = 2e$, we have $Y = -\bar{x}^2 y$, and the inner sum in (4.2) is non-zero, with value p^e, only if $M \equiv N\bar{x}^2 \bmod p^e$. If $p \geq 3$, this congruence has exactly two solutions modulo p^e, leading to the bound $2p^e = 2p^{d/2}$. For $p = 2$, there are four solutions, and the bound is $4 \cdot 2^{d/2}$.

For $d = 2e + 1$, additional work is required. Lemma 7 of [10] gives

$$\left| \sum_{y \bmod p^{e+1}} e^{2\pi i((M - N\bar{x}^2)yp^{-e-1} + N\bar{x}^3 y^2 p^{-1})} \right| \leq \begin{cases} p^{e+1/2} & \text{if } p^e \mid M - \bar{x}^2 N \\ 0 & \text{otherwise.} \end{cases}$$

This leads to the required bounds for odd values of d. □

Lemma 4.7. *The estimate in Proposition* 4.1 *holds with the following choices:*

$$\begin{aligned} f_2(\nu, \mu; e) &= \varphi(2^e)\varphi(2^{e+2})^{-1} t_2(\nu, \mu; e+2), \\ f_3(\nu, \mu; e) &= \varphi(3^e)\varphi(3^{e+1})^{-1} t_3(\nu, \mu; e+1), \\ f_p(\nu, \mu; e) &= t_p(\nu, \mu; e) \qquad \text{if } p \geq 5. \end{aligned}$$

Proof. Clear from the Lemmas 4.5 and 4.6. □

4.2. Weight $\frac{1}{2}$

The estimates of $S_{\pm 1/2}\left(\pm\frac{1}{24}, \pm\frac{1}{24} \mp \mu; c\right)$ mentioned in Subsection 1.3, are based on a formula for $e^{\pi i S(d,c)}$ (with c and d coprime), which Whiteman, [33], attributes to Selberg:

$$\sum_{j \bmod 2c} (-1)^j e^{-\pi i j(3j-1)d/c} \frac{\cos(\pi(j - 1/6)/c)}{\sqrt{3c}} = \begin{cases} e^{\pi i S(d,c)} & \text{if } (d,c) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

for $c, d \in \mathbb{Z}$, $c \geq 1$. This formula is derived from the representation of the eta function of Dedekind $\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)$ as the theta series $\eta(z) =$

Equality (4.3) implies the following relation, which shifts the weight over $\frac{1}{2}$:

$$S_{r+1/2}\Big(n+\frac{1}{24}, m+\frac{1}{24}; c\Big) \qquad (4.4)$$
$$= \sum_{j \bmod 2c} (-1)^j \frac{\cos(\pi(j-1/6)/c)}{\sqrt{3c}} S_r\Big(n, m-\frac{1}{2}j(3j-1); c\Big).$$

This does not seem to lead to explicit bounds of Kloosterman sums for general values of r, n and m.

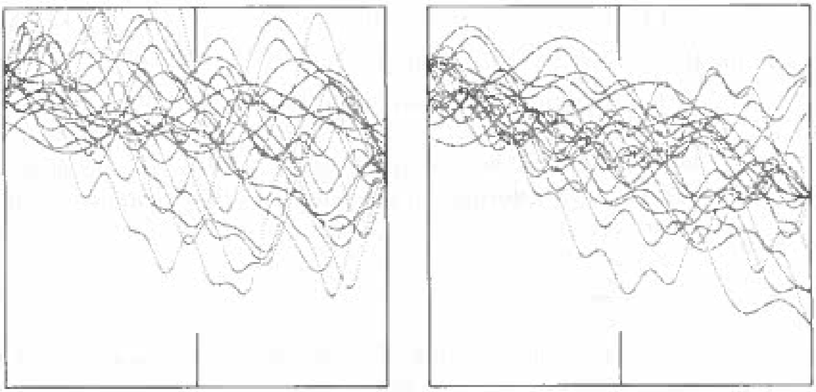

Figure 7: Graphs of $S_r(r/12+1, r/12+1; p)/f_p$ (left) and $S_r(r/12+1, r/12+2; p)/f_p$ (right) for $\frac{1}{2} \leq r \leq \frac{3}{2}$ and primes $p \in [20, 120]$. The lines indicate the window $[-1, 1]$ on the vertical scale at $r = 1$. The bounds f_p come from Proposition 4.1.

If $r = n = 0$, a simplification is possible. In the approach in [25], §123–126, first one uses the 0 in (4.3), to sum over all d modulo c after inserting the definition of $S_0\left(0, m - \frac{1}{2}j(3j-1); c\right)$. This leads to an explicit expression, which reduces to (1.12) if c is a prime number.

Comparison of the Figures 4 and 7 at $r = \frac{1}{2}$ gives an illustration of the difference between the cases $n = 0$ and $n \neq 0$.

5. Derivatives of the Kloosterman-Selberg series

This section discusses the relation between the Kloosterman-Selberg series and Fourier coefficients of Poincaré series, in their dependence on the weight r and the spectral parameter s. It contains the proof of the statements in Part ii) of

5.1. Poincaré series

We consider functions on the upper half-plane $\mathfrak{H}$ that transform under the modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ according to the multiplier system v_r:

$$f\left(\frac{az+b}{cz+d}\right) = v_r\begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{ir\arg(cz+d)} f(z) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{5.1}$$

We start with the functions

$$\mu^m(r,s;z) := y^s e^{2\pi imz} {}_1F_1\left[\begin{matrix} s-r/2 \\ 2s \end{matrix}\middle| 4\pi my\right], \tag{5.2}$$

$$\omega^m(r,s;z) := e^{2\pi imx} W_{\varepsilon r/2,s-1/2}(4\pi|m|y) \tag{5.3}$$

where $m \equiv \frac{r}{12} \bmod 1$, $\varepsilon = \mathrm{sign}(m)$, $m \neq 0$ in (5.3). (See, e.g., [30], (1.1.8) for ${}_1F_1$, and (1.7.1) for $W_{\cdot,\cdot}$.) These functions satisfy (5.1) for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty := \{\pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\} \subset \Gamma$, and are eigenfunctions with eigenvalue $s(1-s)$ under the Casimir operator $C_r := -y^2\partial_x^2 - y^2\partial_y^2 + iry\partial_x$ (in the coordinates $z = x+iy$ on $\mathfrak{H}$).

Poincaré series constitute a way of producing functions on the upper half-plane $\mathfrak{H}$ with a prescribed transformation behavior under the modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. We form the series

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} v_r(\gamma)^{-1} e^{-ir\arg(cz+d)} \mu^m(r,s;\gamma\cdot z). \tag{5.4}$$

For $\mathrm{Re}\, s > 1$ it converges absolutely, uniformly on compact sets in $\mathfrak{H}$, and defines a function $p^m(r,s)$ on $\mathfrak{H}$ which transforms according to (5.1) and is an eigenfunction of C_r with eigenvalue $s(1-s)$.

This Poincaré series has a simpler behavior under C_r than the one introduced by Selberg, see [29] and [11]. On the other hand, it does not determine a square integrable function, except for special values of (r,s).

Any weight $q \equiv r \bmod 2$ can be used with the multiplier system v_r. Here we stick to weight r.

Any reasonable function f satisfying (5.1) has a Fourier expansion with respect to $x = \mathrm{Re}\, z$:

$$f(z) = \sum_{n\equiv r/12 \bmod 1} F_n f(z), \tag{5.5}$$

where $F_n f$, the *Fourier term of order* n, satisfies $F_n f(z+\xi) = e^{2\pi in\xi} F_n f(z)$ for $\xi \in \mathbb{R}$. If f is also an eigenfunction of C_r with eigenvalue $s(1-s)$, then $F_n f$ is a linear combination of the functions in (5.2) and (5.3), for general values of (r,s). A computation shows (see, e.g., [4], Proposition 5.2.9) that if $n \neq 0$,

$$F_n p^m(r,s) = \delta_{m,n}\mu^m(r,s) + c_n^m(r,s)\omega^n(r,s), \tag{5.6}$$

$$c_n^m(r,s) = \frac{\pi^s |n|^{s-1}}{\Gamma(s+\varepsilon(n)r/2)} Z_n^m(r,s), \tag{5.7}$$

$$Z_n^m(r,s) := \sum^{\infty} \frac{\Gamma(2s) S_r(n,m;c)}{c(4\pi^2|mn|)^{s-1/2}} J_{2s-1}^{\varepsilon(nm)}\left(4\pi\sqrt{|mn|}/c\right) \tag{5.8}$$

$$= \sum_{k=0}^{\infty} \frac{(2\pi i)^{2k}(nm)^k}{k!\,(2s)_k} L(n,m;r,s+k). \tag{5.9}$$

We write $\varepsilon(N) = \operatorname{sign}(N)$, and we use the Bessel functions $J_\cdot^1 = J_\cdot$ and $J_\cdot^{-1} = I_\cdot$.

5.2. Meromorphic continuation

The analytic continuation of Poincaré series of Selberg type uses the spectral theory of the operator C_r in the Hilbert space $L^2(\Gamma\backslash\mathfrak{H}, \chi_r)$ of (classes of) functions that satisfy the transformation behavior (5.1) and are square integrable on a fundamental domain for Γ; see [29]. Spectral theory also gives the continuation, as a function of the spectral parameter s, of the Poincaré series given in (5.4); see, e.g., [21], or, in a more general context, [19].

Perturbation theory of linear operators, combined with ideas of Colin de Verdière, [7], leads to the meromorphic continuation in the weight r and the spectral parameter s jointly. In [1], I have done this for the modular case discussed in this paper. The book [4] considers the same question for general cofinite discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$.

Proposition 5.1. *There is a neighborhood U of $(0,12)$ in $\mathbb{C}$ such that for each $\mu \in \mathbb{Z}$ there is a unique meromorphic family of automorphic forms P^μ on $U \times \mathbb{C}$ satisfying*

i) *On a dense set of $(r,s) \in U \times \mathbb{C}$ the function $z \mapsto P^\mu(r,s;z)$ satisfies (5.1) and is an eigenfunction of C_r with eigenvalue $s(1-s)$.*

ii) *For each $(\nu,\mu) \in \mathbb{Z}^2$, there is a meromorphic function $D(\nu,\mu)$ on $U \times \mathbb{C}$ such that $F_n P^\mu(r,s) = \delta_{\nu,\mu}\mu^n(r,s) + D(\nu,\mu;r,s)\omega^n(r,s)$ on a dense subset of $U \times \mathbb{C}$.*

If $r_0 \in (0,12)$, and $\operatorname{Re} s_0 > 1$, the family P^μ is holomorphic at (r_0,s_0), and $P^\mu(r_0,s_0) = p^{m_0}(r_0,s_0)$. The functions $D(\nu,\mu)$ are holomorphic at these points, and $D(\nu,\mu;r_0,s_0) = c_{n_0}^{m_0}(r_0,s_0)$.

Here and in the sequel, I use the convention $n = \nu + \frac{r}{12}$, $m = \mu + \frac{r}{12}$, $n_0 = \nu + \frac{r_0}{12}$, etc. Condition i) implies that the meromorphic functions $(r,s) \mapsto P^\mu(r,s;z)$, with z running through $\mathfrak{H}$, have a common denominator. The dense set in ii) is smaller than that in i), as we have to exclude those (r,s) for which $\mu^n(r,s)$ and $\omega^n(r,s)$ do not form a basis of the space of possible Fourier terms. Note that here automorphic forms are allowed to have complex weight.

The last assertion of the proposition shows that we may call P^μ a meromorphic extension of the Poincaré series; I call it a Poincaré family. The uniqueness implies a functional equation for $s \mapsto -s$, see Section 10.3 of [4].

The restriction to a neighborhood of $(0,12)$ is essential. A multivalued extension to $\mathbb{C}^2 \supset U \times \mathbb{C}$ is possible, but branches along each complex line $\{12k\} \times \mathbb{C}$, $k \in \mathbb{Z}$. Subsection 5.5 gives some indication what happens along $\{0\} \times \mathbb{C}$.

A lot of work is needed to prove this result. Below, I give a proof by reference. For the main ideas, the reader may consult Chapter 1 of [4].

continuation is $(0,12)$. The set $\mathcal{P}$ consists of the only cusp $P = \Gamma \cdot \infty$ of $\Gamma\backslash\mathfrak{H}$. Take $n(r) = \mu + \frac{r}{12}$, $\chi_1 = 1$, and $\ell(r) = r$.) The holomorphy at points (r_0, s_0) is given in Proposition 10.2.12 of [4]. See Section 10.3 of [4] for the Fourier coefficients $D(\nu,\mu)$. Note the normalization of the spectral parameter: $s_{\text{here}} = \frac{1}{2} + s_{(\text{in }[4])}$. □

We conclude from (5.7)–(5.8) that there are meromorphic functions $Z(\nu,\mu) : (r,s) \mapsto \pi^{-s}\left(\varepsilon(n)(\nu + \frac{r}{12})\right)^{1-s} \Gamma\left(s + \varepsilon(n)r/2\right) D(\nu,\mu;r,s)$ on $U \times \mathbb{C}$, that are holomorphic at (r_0,s_0) with $0 < r_0 < 12$ and $\operatorname{Re} s_0 > 1$, with value $Z_{n_0}^{m_0}(r_0,s_0)$.

5.3. Singularities

The singularities of $s \mapsto P^\mu(r_0,s)$, with $0 < r_0 < 12$, on $\operatorname{Re} s > \frac{1}{2}$, are known. They are related to the occurrence of square integrable automorphic forms with the same weight and spectral parameter. But a singularity of P^μ itself need not be visible after restriction to a vertical line. So we have to be careful if we want to catch all singularities occurring at points of $(0,12)\times\mathbb{C}$. I have no results concerning singularities at points (r_0,s_0) with $r_0 \notin \mathbb{R}$.

From (1.8), (1.7) and (5.8), it follows that

$$Z(\nu,\mu;r,s) = Z\left(-\nu-1,-\mu-1;12-r,s\right). \tag{5.10}$$

So it suffices to study singularities at (r_0,s_0) with $0 < r_0 \leq 6$ and $\operatorname{Re} s_0 > \frac{1}{2}$.

Consider first one-parameter Poincaré families $\dot{P}^\mu(r) : s \mapsto P^\mu(r,s)$, with $0 < r \leq 6$. For the modular case considered in this paper, we know that there is only one eigenvalue of C_r smaller than $\frac{1}{4}$. It is $\frac{r}{2}(1-\frac{r}{2})$, corresponding to the square integrable automorphic form $\eta_r(z) := y^{r/2}\eta(z)^{2r}$; the multiplicity is one. (See Theorem 2.15 in [1, Part III] for the absence of exceptional eigenvalues of continuous series type.)

For general values of (r,s) with r real and $\frac{1}{2} \leq \operatorname{Re} s < \frac{1}{2}$, singularities of $\dot{P}^\mu(r)$ at s are in direct correspondence with eigenvalues of the Casimir operator. But for $s = 1 - \frac{r}{2}$, corresponding to the eigenvalue of η_r, there is a complicating factor: μ^m and ω^m need not provide us with a basis of the space of Fourier terms. That complicates the computations and the results. Proposition 11.3.9 of [4] shows that the only case in which $\dot{P}^\mu(r)$ is singular at a point of $\operatorname{Re} s > \frac{1}{2}$ is $0 < r < 1$. In this case, there may be a pole of order at most one at $s = 1 - \frac{r}{2}$. For the residue, I have found conflicting results in the literature. I have confidence in the result given in Part (ii) of Theorem 4.1 in [13]:

$$\lim_{s \to 1-r/2} \left(s - 1 + \frac{r}{2}\right) P^\mu(r,s) = p_\mu(r) N_\eta(r)^{-1} \eta_r. \tag{5.11}$$

(To apply [13], we use the following identifications: $\mathbf{M}^\eta(\xi, 2s-1, p(z), \varphi_r) = \mathbf{M}^\eta(2s-1,\varphi_r,\chi) = \Gamma(2s)^{-1} P^\mu(r,s;z)$, $z \mapsto p(z)$ is the standard lift from $\mathfrak{H}$ to the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, $\lambda = \mu + \frac{r}{12}$. The corresponding Fourier coefficient $c(\eta, f)$ is equal to $(4\pi|m|)^{-r/2} p_\mu(r)$, and η_r spans the relevant space of square integrable modular forms. Use also that $\omega^m(r,(r-1)/2;iy) = (4\pi my)^{r/2}e^{-2\pi my}$ if $m > 0$.)

Note that $p_\mu(r)$ may vanish for certain $r \in (0,1)$; it vanishes identically if $\mu < 0$. In these cases, $\dot{P}^\mu(r)$ is holomorphic on $\mathrm{Re}\, s > \frac{1}{2}$.

For the Fourier coefficient of order n, we find the same pole behavior on $\mathrm{Re}\, s > \frac{1}{2}$, and for $0 < r < 1$:

$$\lim_{s\to 1-r/2} \left(s - 1 + \frac{r}{2}\right) D(\nu,\mu;r,s) = (4\pi|n|)^{-r/2} p_\mu(r) p_\nu(r) N_\eta(r)^{-1}. \qquad (5.12)$$

Next we ask whether the family P^μ, depending on two variables, can have a singularity at other points (r_0, s_0), with $0 < r_0 \leq 6$, $\mathrm{Re}\, s_0 > \frac{1}{2}$. The restriction $\dot{P}^\mu(r_0)$ to the line $\{r_0\} \times \mathbb{C}$ exists. So the polar divisor of P^μ projects onto a neighborhood of r_0. This neighborhood contains other real points r_n, and there exists s_n such that P^μ is singular at (r_n, s_n), and $(r_n, s_n) \to (r_0, s_0)$. The set of points at which P^μ is indeterminate has codimension 2. So the restriction $\dot{P}^\mu(r_n)$ is singular at s_n for most n. These (r_n, s_n) satisfy $0 < r_n < 1$ and $s_n = 1 - \frac{1}{2}r_n$. (Chapter 12 of [4] gives a more detailed discussion of questions of this type.) We have obtained the following result:

Proposition 5.2. *Let $r_0 \in (0,6]$, $\mathrm{Re}\, s_0 > \frac{1}{2}$, $\mu,\nu \in \mathbb{Z}$.*

If P^μ is singular at (r_0, s_0), then $\mu \geq 0$, $r_0 \in (0,1)$, $s_0 = 1 - \frac{1}{2}r_0$, and $(r,s) \mapsto \left(s - 1 + \frac{r}{2}\right) P^\mu(r,s)$ is holomorphic at (r_0,s_0), with value given in (5.11).

If $D(\nu,\mu)$ is singular at (r_0,s_0), then $\mu \geq 0$, $\nu \geq 0$, $r_0 \in (0,1)$, $s_0 = 1 - \frac{1}{2}r_0$, and $(r,s) \mapsto \left(s - 1 + \frac{r}{2}\right) D(\nu,\mu;r,s)$ is holomorphic at (r_0,s_0), with value given in (5.12).

In the transition from $D(\nu,\mu)$ to $Z(\nu,\mu)$, the factor $\Gamma\left(s + \frac{1}{2}\varepsilon(n)r\right)$ may cause additional singularities if $\nu < 0$. Let us consider first the case $s = \frac{1}{2}r$, $1 < r \leq 6$.

If $\mu \geq 0$, Part (i) of Theorem 4.1 in [13] implies that $P^\mu(r, r/2)$ corresponds to a cusp form in the Lie algebra module generated by a holomorphic cusp form. Hence all Fourier terms of negative order vanish. This cancels the singularity of $\Gamma(s - r/2)$.

If $\mu < 0$, we still have $P^\mu(r, r/2; z) = y^{r/2}h(z)$, with h a holomorphic automorphic form for the multiplier system v_r. As it has the form $h(z) = e^{2\pi i m z} + o(1)$, the other Fourier terms of negative order still vanish, and the singularity of the gamma factor is cancelled.

Let $1 < r \leq 6$, $\frac{1}{2} < s < \frac{1}{2}r$, $s = \frac{1}{2}r - c$, $c \in \mathbb{Z}$. The automorphic form $(\mathbf{E}^-)^c P^\mu(r,s)$, of weight $r - 2c$, corresponds to a holomorphic automorphic form of the type considered for $c = 0$. The Maass operator $\mathbf{E}^-$ does not change the type of the Fourier expansion, so the same reasoning goes through. (See, e.g., 2.2.3–4, 4.4-5 in [4] for the definition and properties of $\mathbf{E}^-$.)

Proposition 5.3. *Let $r_0 \in (0,12)$, $\mathrm{Re}\, s_0 > \frac{1}{2}$, $\mu,\nu \in \mathbb{Z}$. A singularity of $Z(\nu,\mu)$ at (r_0,s_0) occurs only in the following cases:*

a) $r_0 \in (0,1)$, $s_0 = 1 - \frac{1}{2}r_0$, $\nu \geq 0$, $\mu \geq 0$.

The function $(r,s) \mapsto \left(s - 1 + \frac{r}{2}\right) Z(\nu,\mu;r,s)$ is holomorphic at (r_0,s_0) with

b) $r_0 \in (11,12)$, $s_0 = \frac{1}{2}r_0 - 5$, $\nu \le -1$, $\mu \le -1$.
The function $(r,s) \mapsto \left(s + 5 - \frac{1}{2}r_0\right) Z(\nu,\mu;r,s)$ *is holomorphic at* (r_0,s_0) *with value* $\pi^{-1}2^{r_0-12}p_{-\mu-1}(12-r_0)p_{-\nu-1}(12-r_0)N_\eta(12-r_0)^{-1}$.

5.4. Extension of the Kloosterman-Selberg series

Proposition 2.1 gives information on $L_k(n,m;r,s)$ as a function of s. Here I consider its dependence on r and s jointly. I have not succeeded in proving that it is the restriction of a meromorphic function of (r,s).

Part ii) of Proposition 2.1 is a consequence of the Propositions 5.4 and 5.5.

For technical reasons, I look not only at the derivatives of the Kloosterman-Selberg series with respect to r, but also with respect to s. The integers ν and μ are fixed in this subsection; I omit them from the notation. The convention $n = \nu + \frac{r}{12}$, $m = \mu + \frac{r}{12}$ stays valid. $\mathbb{C}_a$ denotes the right half-plane $\{\, s \in \mathbb{C} : \operatorname{Re} s > a \,\}$, and J denotes the interval $(0,12)$.

For $k,j \in \mathbb{Z}_{\ge 0}$, $r \in J$, and $\operatorname{Re} s > 1 + \frac{k}{2}$, the following series converges absolutely:

$$\begin{aligned}\Delta(k,j;r,s) &= \Delta(k,j,\nu,\mu;r,s) \qquad (5.13)\\ &:= \sum_{c=1}^{\infty} (\log c)^j\, c^{-2s} \sum_{d \bmod c}^{*} S(d,c)^k e^{2\pi i r S(d,c) + 2\pi i(\nu a + \mu d)/c}.\end{aligned}$$

This defines $\Delta(k,j)$ as a continuous function on $J \times \mathbb{C}_{1+k/2}$, which is holomorphic in s, and satisfies $\partial_s \Delta(k,j) = -2\Delta(k,j+1)$. On $J \times \mathbb{C}_{1+(k+1)/2}$, we have $\partial_r \Delta(k,j) = 2\pi i \Delta(k+1,j)$. Note that $L_k(n,m;r,s) = \Delta(k,0;r,s)$.

In the following result, the set U is an open set in $\mathbb{C}$ containing J, such that the Poincaré families are defined on $U \times \mathbb{C}$.

Proposition 5.4. *Let* $\nu,\mu \in \mathbb{Z}$. *There is an analytic set* $S = S(\nu,\mu) \subset U \times \mathbb{C}$, *such that the following holds:*

i) *The intersection* $(\{r\} \times \mathbb{C}) \cap S$ *is discrete for each* $r \in J$, *and is contained in*

$$\begin{cases} \{s \in \mathbb{C} : \operatorname{Re} s \le \frac{1}{2}\} \cup \{1 - \frac{r}{2}\} & \text{if } 0 < r < 11,\\ \{s \in \mathbb{C} : \operatorname{Re} s \le \frac{1}{2}\} & \text{if } 1 \le r \le 11,\\ \{s \in \mathbb{C} : \operatorname{Re} s \le \frac{1}{2}\} \cup \{\frac{r}{2} - 5\} & \text{if } 11 < r < 12.\end{cases}$$

ii) *For all integral* $k,j \ge 0$, *there are smooth functions* $\Delta(k,j) = \Delta(k,j,\nu,\mu)$ *on* $(J \times \mathbb{C}) \setminus S$ *satisfying the following conditions:*

a) *On* $J \times \mathbb{C}_{1+k/2}$ *they coincide with the functions in* (5.13).

b) *For each* $r \in J$, *the maps* $s \mapsto \Delta(k,j;r,s)$ *are the restrictions of meromorphic functions on* $\mathbb{C}$.

c) *On* $(J \times \mathbb{C}) \setminus S$:

$$\partial_r \Delta(k,j;r,s) = 2\pi i \Delta(k+1,j;r,s), \qquad (5.14)$$

Proof. To define the set S, first consider the set T of points in $U \times \mathbb{C}$ at which the function $(r,s) \mapsto Z(\mu,\mu;r,s)$ is singular. So T is an analytic subset of $U \times \mathbb{C}$, not intersecting $J \times \mathbb{C}_1$. Its only points (r,s) with $r \in J$ and $\mathrm{Re}\, s > \frac{1}{2}$, are $\left(r, 1-\frac{r}{2}\right)$ with $r \in (0,1)$, and $\left(r, \frac{r}{2}-5\right)$ with $r \in (11,12)$; see Proposition 5.3. This set T intersects lines $\{r\} \times \mathbb{C}$ discretely. We define S to be the union of the translates $T - \frac{1}{2}n$, $n \in \mathbb{Z}_{\geq 0}$, and the set $U \times \frac{1}{2}\mathbb{Z}_{\leq 0}$. In this way, S satisfies condition i), and also $S - \frac{1}{2} \subset S$.

In Part ii), we need not prove smoothness. It suffices to prove continuity, meromorphy in s, and partial differentiability with respect to r.

The proof of Part ii) consists of two nested induction loops. The outer loop is over $a \in \frac{1}{2}\mathbb{Z}_{\leq 2}$, starting at $a = 1$ and going down with steps $\frac{1}{2}$. The inner induction is over $j \in \mathbb{Z}_{\geq 0}$. The induction statement in the inner loop is:

For all $k \in \mathbb{Z}_{\geq 0}$, there is a continuous function $\Delta(k,j)$ on $\left(J \times \mathbb{C}_{a+k/2}\right) \setminus S$, with the following properties:

(h1) *On $J \times \mathbb{C}_{1+k/2}$, it is given by the series in* (5.13).

(h2) *For each $r \in J$, the function $s \mapsto \Delta(k,j;r,s)$ extends meromorphically to $\mathbb{C}_{a+k/2}$.*

(h3) *$\Delta(k,j)$ is partial differentiable with respect to r on $\left(J \times \mathbb{C}_{a+(k+1)/2}\right) \setminus S$, and satisfies* (5.14) *on this region.*

In the outer loop we show

(h4) *$\Delta(k,j)$ satisfies* (5.15) *on $\left(J \times \mathbb{C}_{a+k/2}\right) \setminus S$ for all $k, j \geq 0$.*

These statements are true for $a = 1$ and all $j \geq 0$.

Let $a \in \frac{1}{2}\mathbb{Z}$, $a < 1$, be given, and suppose that (h1)–(h4) have been shown for all larger values of a. Suppose that, in the inner loop, (h1)–(h3) have been demonstrated for all smaller values of j.

We define, for $h \in \mathbb{Z}_{\geq 0}$, $\beta_h(u,s) := \dfrac{(2\pi i)^{2h} u^h}{h!\,(2s)_h}$. We use the following relation, valid for $(r,s) \in J \times \mathbb{C}_{1+h/2}$ (see (5.9)):

$$\partial_r^k \partial_s^j Z(\nu,\mu;r,s) \tag{5.16}$$
$$= \sum_{h=0}^{\infty} \sum_{p=0}^{k} \sum_{q=0}^{j} \binom{k}{p} \binom{j}{q} (2\pi i)^p (-2)^q \Delta(p,q;r,s+h) \left(\partial_r^{k-p} \partial_s^{j-q} \beta_h(nm,s)\right).$$

The left-hand side is meromorphic on $U \times \mathbb{C}_a$, with singularities contained in $T \subset S$, so it is continuous on $(J \times \mathbb{C}_a) \setminus S$, meromorphic in s, and partial differentiable with respect to r for $\mathrm{Re}\, s > a + \frac{k+1}{2}$. The same properties are enjoyed by all terms in the right-hand side, except by the term with $p = k$ and $q = j$. All terms in the right-hand side, except finitely many, satisfy $a + \frac{k}{2} + h \geq 1 + \frac{p}{2}$. We insert the series in (5.13) into the infinite sum of these terms, which are in the region of absolute convergence of the series in (5.13). Inspection of β_h and its derivatives shows that the infinite sum is also continuous on $\left(J \times \mathbb{C}_{a+k/2}\right) \setminus S$, meromorphic in s, and partial differentiable with respect to r if $\mathrm{Re}\, s > a + \frac{k+1}{2}$. (In fact, the poles arise only from the derivatives of $\frac{1}{(2s)_h}$. That is the reason why we the continuous continuation of $\Delta(k,j)$ to $\left(J \times \mathbb{C}_{a+k/2}\right) \setminus S$, and meromorphy in the variable s. (The meromorphy ensures that the continuation is unique.) So conditions (h1) and (h2) are satisfied.

The partial derivative $\partial_r \Delta(k,j)$ at a point (r,s) with $a + \frac{k}{2} < \operatorname{Re} s \leq a + \frac{k+1}{2}$, can be computed with relation (5.16). We use relation (5.16) also with k replaced by $k+1$, and use the induction hypothesis, to see that $\Delta(k,j)$ satisfies (h3).

After we have completed the inner induction loop on j, we use (5.16) with j replaced by $j+1$. Now we prove (h4) in a similar way. □

To complete the proof of Proposition 2.1, we look at the singularities in the region $\frac{1}{2} < \operatorname{Re} s \leq 1$.

Proposition 5.5. *Let $\nu, \mu \in \mathbb{Z}$, $k, j \in \mathbb{Z}_{\geq 0}$. The function $\Delta(k,j,\nu,\mu)$ can have a singularity at (r,s) with $r \in (0,12)$, $\frac{1}{2} < \operatorname{Re} s \leq 1$, only in the following two cases:*
a) *A singularity at $\left(r, 1 - \frac{r}{2}\right)$, with $0 < r < 1$.*

$$\lim_{s \to 1 - r/2} \left(s + \frac{r}{2} - 1\right)^{k+j+1} \Delta(k,j,\nu,\mu;r,s) = \frac{(j+k)!\, p_\nu(r) p_\mu(r)}{\pi 2^{r+k+j} (-2\pi i)^k N_\eta(r)}.$$

b) *A singularity at $\left(r, \frac{r}{2} - 5\right)$, with $11 < r < 12$.*

$$\lim_{s \to r/2 - 5} \left(s + 5 - \frac{r}{2}\right)^{k+j+1} \Delta(k,j,\nu,\mu;r,s)$$
$$= \frac{(j+k)!\, p_{-\nu-1}(12-r) p_{-\mu-1}(12-r)}{\pi 2^{12+k+j-r} (2\pi i)^k N_\eta(12-r)}.$$

Proof. We consider $0 < r < 1$; the other case is similar.

The reasoning in the proof of the previous proposition and (5.16) show that it suffices to consider the singularity of $(2\pi i)^{-k}(-2)^{-j} \partial_r^k \partial_s^j Z(\nu,\mu;r,s)$. A principal part of $Z(\nu,\mu;r,s)$ at $(r_0, 1 - r_0/2)$ is of the form $\frac{a(r,s)}{s + r/2 - 1}$, where a is a holomorphic function with value $\pi^{-1} 2^{-r_0} p_\mu(r_0) p_\mu(r_0) N_\eta(r_0)^{-1}$ at $(r_0, 1 - r_0/2)$. Now we can carry out $\partial_r^k \partial_s^j$ as a completely complex differentiation. The worst that can happen is a factor $(s + r/2 - 1)^{j+k+1}$ in the denominator. The proposition follows. □

5.5. Weight zero

The restriction of the weight to the interval $(0,12)$ in the Subsections 5.2–5.4 is essential. For $r \in 12\mathbb{Z}$, the operator C_r has not only a discrete, but also a continuous spectrum. This make application of linear perturbation theory more difficult around $r = 0$, than around points of the interval $(0,12)$.

In [2] and [5], the continuation of Poincaré series on a neighborhood of $\{0\} \times \mathbb{C}$ is essential to obtain a meromorphic continuation of $L_k(n,m;0,s)$, and the assertions in Part i) of Proposition 2.1.

I used families of automorphic forms E^ν defined on $U_0 \times \mathbb{C}$, where U_0 is a

Poincaré families:

$$E^{\mu}(r,s) = P^{\mu}(r,s) - \frac{v(r,s)D(0,\mu;r,s)}{1+v(r,s)D(0,0;r,s)}P^{0}(r,s), \qquad (5.17)$$

$$v(r,s) := \left(\frac{\pi r}{3}\right)^{s} \frac{\Gamma(1-2s)}{\Gamma(1-s-\frac{r}{2})}. \qquad (5.18)$$

Note that the factor $\left(\frac{\pi r}{3}\right)^{s}$ makes v singular along the line $\{0\}\times\mathbb{C}$ (not meromorphic). When we reverse relation (5.17), we see that P^{μ} has a multivalued meromorphic extension around $r=0$.

Nevertheless, the family E^{ν} has a meromorphic extension to the line $\{0\}\times\mathbb{C}$. The restriction $s\mapsto E^{\nu}(0,s)$ exists, and is given by an Eisenstein or Poincaré series for $\mathrm{Re}\, s>1$. A peculiarity of the family E^{ν} are its singularities at $\left(0,\frac{\ell}{2}\right)$ for integral $\ell>2$. This leads to the singularities described in Part i) of Proposition 2.1.

References

[1] Bruggeman, R.W., Modular forms of varying weight, I–III. Math. Z. 190 (1985), 477–495; ibid. 192 (1986), 297–328; J. Reine Angew. Math. 371 (1986), 144–190.

[2] — Eisenstein series and the distribution of Dedekind sums. Math. Z. 202 (1989), 181–198.

[3] — Dedekind sums and Fourier coefficients of modular forms. J. Number Theory 36 (1990), 289–321.

[4] — Families of automorphic forms (Monogr. Math. 88). Birkhäuser, Basel 1994.

[5] — Dedekind sums for Hecke groups. Acta Arith. 71 (1995), 11–46.

[6] Bruggeman, R.W., Miatello, R.J., Estimates of Kloosterman sums for groups of real rank one. Duke Math. J. 80 (1995), 105–137.

[7] Colin de Verdière, Y., Pseudo-Laplaciens II. Ann. Inst. Fourier (Grenoble) 33(2) (1983), 87–113.

[8] Dedekind, R., Erläuterungen zu den Fragmenten XXVIII. In: Riemann, B., Gesammelte mathematische Werke und wissenschaftlicher Nachlass (ed. by H. Weber), 466–478, reprint, Dover Publ., New York 1953.

[9] Deshouillers, J.-M., Iwaniec, H., Kloosterman sums and Fourier coefficients of cusp forms. Invent. Math. 70 (1982), 219–288.

[10] Estermann, Th., On Kloosterman's sum. Mathematika 8 (1961), 83–86.

[11] Goldfeld, D., Sarnak, P., Sums of Kloosterman Sums. Invent. Math. 71 (1983), 243–250.

[12] Iwaniec, H., Introduction to the spectral theory of automorphic forms (Bibl. Rev. Mat. Iberoamericana). Rev. Mat. Iberoam., Madrid 1995.

[13] Kisbye, N.P., Miatello, R.J., On the residues and special values of Poincaré series. Indag. Math. (N.S.) 9 (1998), 35–53.

[14] Kloosterman, H.D., On the representation of numbers in the form $ax^2+by^2+cz^2+dt^2$. Acta Math. 49 (1926), 407–464.

[15] — Asymptotische Formeln für die Fourierkoeffizienten ganzer Modulformen. Abh.

[16] Kuznetsov, N.V., The Petersson conjecture for parabolic forms of weight zero and the conjecture of Linnik. Sums of Kloosterman sums (Russian). Mat. Sb. 111 (1980), 334–383.

[17] Linnik, Yu.V., Additive problems and eigenvalues of the modular operators. In: Proc. Internat. Congr. Mathematicians (Stockholm, 1962), 270–284. Inst. Mittag-Leffler, Djursholm 1963.

[18] Malyshev, A.V., Generalization of Kloosterman sums and their estimates (Russian). Vestnik Leningrad. Univ. 15(13) (1960), 59–75.

[19] Miatello, R., Wallach, N.R., Automorphic forms constructed from Whittaker vectors. J. Funct. Anal. 86 (1989), 411–487.

[20] Myerson, G., Dedekind sums and uniform distribution. J. Number Theory 28 (1988), 233–239.

[21] Neunhöffer, H., Über die analytische Fortsetzung von Poincaréreihen. Sitzungsb. Heidelberger Akad. Wiss. Math.-Natur. Kl. (1973), 33–90.

[22] Patterson, S.J., The asymptotic distribution of Kloosterman sums. Acta Arith. 79 (1997), 205–219.

[23] Petersson, H., Über die Entwicklungskoeffizienten der automorphen Formen. Acta Math. 58 (1932), 169–215.

[24] — Zur analytischen Theorie der Grenzkreisgruppen, I. Math. Ann. 115 (1937), 23–67.

[25] Rademacher, H., Topics in analytic number theory (Grundlehren Math. Wiss. 169). Springer, New York 1973.

[26] Rademacher, H., Grosswald, E., Dedekind sums. The Mathematical Association of America, Washington 1972.

[27] Salié, H., Über die Kloostermanschen Summen $S(u, v; q)$. Math. Z. 34 (1932), 91–109.

[28] — Zur Abschätzung der Fourierkoeffizienten ganzer Modulformen. Math. Z. 36 (1933), 263–278.

[29] Selberg, A., On the estimation of Fourier coefficients of modular forms. In: Proc. Symp. Pure Math. VIII, 1–15. Amer. Math. Soc., Providence 1965.

[30] Slater, L.J., Confluent hypergeometric functions. Cambridge University Press, New York 1960.

[31] Vardi, I., A relation between Dedekind sums and Kloosterman sums. Duke Math. J. 55 (1987), 189–197.

[32] Weil, A., On some exponential sums. Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 204–207.

[33] Whiteman, A.L., A sum connected with the series for the partition function. Pacific J. Math. 6 (1956), 159–176.

[34] Yoshida, E., On the order of growth of the Kloosterman zeta function. J. Math. Soc. Japan 44 (1992), 53–73.