Definition 5.7. The space $M_w(\Gamma)$ of holomorphic automorphic forms for Γ with weight $w \in \mathbb{Z}$ consists of the holomorphic functions F on $\mathcal{X}$ that satisfy for all $\gamma \in \Gamma$

$$F\big(\gamma \cdot (z,u)\big) \;=\; j\big(\gamma,(z,u)\big)^w\, F(z)\,. \tag{5.17}$$

Proposition 5.8. *Let $h \in 2\mathbb{Z}$. The linear map Ψ_h in Proposition 5.6 gives a bijection*

$$\Big\{ f \in \mathbf{A}\big(\Gamma;\psi[-h-3,1]\big)_{h,0} \;:\; R(\mathbf{Z}_{31}) f = 0 \Big\} \longrightarrow M_{-h/2}(\Gamma)\,. \tag{5.18}$$

If these spaces are non-zero then $h \in 2\mathbb{Z}_{\leq 0}$. The space $M_0(\Gamma)$ consists of the constant functions. [1]

If $h \leq -2$, the corresponding Fourier expansions are of the following form

$$\begin{aligned} f^{\mathbf{c}} \;&=\; a_0^{\mathbf{c}}\; {}^h\varphi^0_{0,0}(-h/2-2)\\ &\quad + \sum_{\ell \in \frac{\sigma(\mathbf{c})}{2}\mathbb{Z}_{\geq 1}} \;\sum_{c \bmod 2\ell} a^{\mathbf{c}}_{\ell,c}\, \omega^{0,0}_{\ell,c,h+3}\big(\tfrac{h}{2}, -\tfrac{h}{2}-2\big)\,, \end{aligned} \tag{5.19}$$

$$\begin{aligned} (\Psi_h f^{\mathbf{c}})(z,u) \;&=\; a_0^{\mathbf{c}} + \sum_{\ell \in \frac{\sigma(\mathbf{c})}{2}\mathbb{Z}_{\geq 1}} \;\sum_{c \bmod 2\ell} a^{\mathbf{c}}_{\ell,c}\, \frac{\ell^{-1/4}\, e^{\pi i \ell z}}{\sqrt{\pi}\,(2\pi\ell)^{h/4}}\\ &\qquad \cdot \sum_{k\in\mathbb{Z}} e^{\frac12 \pi \ell (c/\ell+2k)^2}\, e^{-\pi\ell\left(u+c/\ell+2k\right)^2}\,. \end{aligned} \tag{5.20}$$

Proof. Proposition 5.6 gives a bijection between $M_{-h/2}(\Gamma)$ and the space of $f \in C^\infty(G)_{h,0}$ that are left-invariant under Γ and satisfy $S_1^3 f = R(\mathbf{Z}_{31}) f = 0$. We express the function f in terms of the holomorphic function $F = \Psi_h f$, and apply the generators C and Δ_3 of $ZU(\mathfrak{g})$ to f. This gives

$$Cf \;=\; \Big(\frac{h^2}{3} + 2h\Big) f\,, \qquad \Delta_3 f \;=\; \frac{1}{9} h(h+12)(h+6)\, f\,. \tag{5.21}$$

These eigenvalues correspond to the character $\psi[-h-3,1]$ of $ZU(\mathfrak{g})$. So f is in the space $\mathbf{A}^{\mathrm{u}}\big(\Gamma;\psi[-h-3,1]\big)_{h,0}$. See [6, §26b].

All Fourier terms $f^{\mathbf{c}}$ have to satisfy $S_1^3 f^{\mathbf{c}} = 0$. This has drastic consequences for the non-zero Fourier terms. By Proposition 3.14 all generic abelian Fourier terms have to vanish.

For each cusp $\mathbf{c}$ the Fourier term $\mathbf{F}_0 f^{\mathbf{c}}$ must be in a principal series module or in a logarithmic module. In (3.78) we see the possibility ${}^h\varphi^0_{0,0}(-h/2-2) = t^{-h/2}\; {}^h\Phi^0_{0,0}$, which corresponds to the constant function 1 on $\mathcal{X}$. A direct computation shows that the logarithmic modules cannot contribute.

Next we look for elements of $\mathcal{F}^{\psi[-h-3,1]}_{\ell,c,d;h,0}$ in the kernel of S_1^3. It suffices to consider $\mathcal{M}^{\psi[-h-3,1]}_{\mathbf{n}}$ and $\mathcal{W}^{\psi[-h-3,1]}_{\mathbf{n}}$. We inspect Figures 4.35–4.46.

We find four cases where S_1^3 vanishes in a one-dimensional K-type: Figures 4.37, 4.39, 4.42 and 4.44. In those cases the vanishing occurs in $\mathcal{W}^{\psi}_{\mathbf{n}}$.

a) Figures 4.42 and 4.44 give $\ell > 0$, $m_0(j_l) \geq 0 > m_0(j_+)$. We read off that $j_+ = j_l + 3$. The K-type gives $h = 2j_l$.

   Table 4.12 gives $0 \leq \nu_l \leq -j_l - 2$ and $0 \leq m_0(j_l) < -\frac{j_l+\nu_l}{2}$. The relation $j_+ = j_l + 3$ is equivalent to $j_l + \nu_l = -2$. This implies that $h = 2j_l \leq -4$.

[1] Correction Prop. 5.8, proof streamlined.

b) Figures 4.37 and 4.39 give $\ell > 0$, $m_0(j_+) \geq 0 > m_0(j_r)$. We read off that $j_r = j_+ + 3$. The K-type gives $h = 2j_+$.

Table 4.12 gives $1 \leq \nu_l \leq -j_l$, $m_0(j_l) \geq 0$, and $-\frac{j_l+\nu_l}{2} \leq m_0(j_l) < \frac{\nu_l - j_l}{2}$. The relation $j_r = j_+ + 3$ is equivalent to $\nu_l = 1$. Since $j_l \equiv \nu_l = 1 \bmod 2$, this gives $j_l = -1$. The point $(j_l, \nu_l) = (-1, 1)$ is equal to (j_+, ν_+), and we get $h = 2j_+ = -2$.

In this way we see that $h \leq -2$ is the condition to have non-abelian terms with $\omega^{0,0}_{\ell,c,h+3}(h/2, -h/2-2)$ in the Fourier expansion. The non-abelian terms form a convergent series of the form considered in Proposition 5.5. Hence this part of the expansion has exponential decay at all cusps. Together with the constant term this gives polynomial growth of all $f^{\mathbf{c}}$, and $f \in \mathbf{A}(\Gamma; \psi[-h-3, 1])$.

The Fourier expansions of f now have the form indicated in (5.19). We have to carry out the transformation to Ψ_h in (5.20), using (3.88), (A.17), (2.52), and Table 2.3. See [6, §26c]. □

5.2. Families of automorphic forms

Eisenstein series and Poincaré series occur in families that are parametrized by ν, so that the Fourier term are meromorphic functions of ν. To formulate this in terms of Fourier coefficients we have to choose basis families. The families in Proposition 5.10 were convenient in our paper [5].

Definition 5.9. Let U be a connected open set in $\mathbb{C}$, let $j \in \mathbb{Z}$ and let τ^h_p be a K-type such that $|h-2j| \leq 3p$. A holomorphic family of automorphic forms for (j, h, p) on U is an element $f \in C^\infty(U \times G)$ such that $g \mapsto f(\nu, g)$ is an element of $\mathbf{A}^!(\Gamma, \psi[j, \nu])_{h,p,p}$ for each $\nu \in U$, and such that $\nu \mapsto f(\nu, g)$ is holomorphic on U for each $g \in G$. A meromorphic family of automorphic forms has, locally in ν, the form $(\nu, g) \mapsto \frac{1}{\varphi(\nu)} f(\nu, g)$ where f is a holomorphic family and φ is a non-zero holomorphic function.

Remarks. (1) The restriction to automorphic forms with moderate exponential growth is non-essential. Prescribing the first spectral parameter j, the K-type τ^h_p, and the highest weight p in the K-type is practical. One obtains more general families as a $U(\mathfrak{k})$-linear combination of families of this type.

(2) Meromorphically continued Eisenstein series are examples.

Families of modules of automorphic forms. Let $\nu \mapsto f(\nu)$ be such a family of automorphic forms. For an open dense subset $U_0 \subset U$ the spectral pairs (j, ν) with $\nu \in U_0$ correspond to simple parametrization. The $(\mathfrak{g}, K)$-module $M(\nu)$ generated by $f(\nu)$ is isomorphic to $H^{\xi_j,\nu}_K$ for $\nu \in U_0$. On $U_0 \cup \{0\}$ we can chose a holomorphic family of isomorphisms $M(\nu) \to H^{\xi_j,\nu}_K$. At $\nu = 0 \notin U_0$ we have to deal with the possibilities of logarithmic modules in Proposition 3.29.

Now we take $\nu_0 \in U$ such that (j, ν_0) corresponds to integral parametrization. At $\nu = \nu_0 \in U_0$ the structure of $M(\nu)$ changes in an essential way. It may become reducible, and families in some K-types in $M(\nu)$ may have a zero at ν_0.

Fourier terms. The Fourier term operators are given by integration over compact sets. Hence if f is a holomorphic family of automorphic forms, then $\nu \mapsto \mathbf{F}_{\mathcal{N}} f(\nu)^{\mathbf{c}}$ is a holomorphic family of elements in the modules of type $\mathcal{F}^{\psi[j,\nu]}_{\mathcal{N}}$.