PERIOD FUNCTIONS FOR VECTOR-VALUED MAASS CUSP FORMS OF REAL WEIGHT, WITH AN APPLICATION TO JACOBI MAASS CUSP FORMS

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Abstract. For vector-valued Maass cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ with real weight $k \in \mathbb{R}$ and spectral parameter $s \in \mathbb{C}$, $\mathrm{Re}\, s \in (0,1)$, $s \not\equiv \pm k/2 \bmod 1$, we propose a notion of vector-valued period functions, and we establish a linear isomorphism between the spaces of Maass cusp forms and period functions by means of a cohomological approach. The period functions are a generalization of those for the classical Maass cusp forms, being solutions of a finite-term functional equation or, equivalently, eigenfunctions with eigenvalue 1 of a transfer operator deduced from the geodesic flow on the modular surface. We apply this result to deduce a notion of period functions and related linear isomorphism for Jacobi Maass forms of weight $k + 1/2$ for the semi-direct product of $\mathrm{SL}_2(\mathbb{Z})$ with the integer points $\mathsf{Hei}(\mathbb{Z})$ of the Heisenberg group.

1. Introduction

For several hyperbolic orbisurfaces $\Gamma\backslash\mathfrak{H}$, with $\mathfrak{H}$ denoting the hyperbolic plane and Γ being a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ acting by fractional linear transformation on $\mathfrak{H}$, notions of period functions for Maass forms and associated isomorphisms have been established in the course of the last years. For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, which is the seminal example, this has been achieved in combination of work by E. Artin [1], Series [35], Mayer [23, 24], Lewis [18], Bruggeman [4], Chang–Mayer [10], and Lewis–Zagier [19, 20]. Alternative proofs are given in [8, 25], and most recently, by combination of [26, 28–30].

The variant of these proofs most relevant for our work proceeds roughly as follows, applying to Maass cusp forms. See also the survey [32]. The space of Maass cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ with spectral parameter s is shown to be linear isomorphic to the space of parabolic 1-cohomology of $\mathrm{SL}_2(\mathbb{Z})$ with module being the vector space of smooth, semi-analytic vectors of the principal series representation with spectral parameter s. The cocycle classes can be characterized by real-analytic, rapidly decaying solutions of a rather simple functional equation on $(0,\infty)$ that depends on s. In this way, Maass cusp forms with spectral parameter s are seen to be linear isomorphic to real-analytic functions on $(0,\infty)$ or, equivalently, holomorphic functions on $\mathbb{C} \setminus (-\infty, 0]$ that satisfy the s-dependent functional equation

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and certain decay properties at boundaries. This isomorphism is given by an integral transform relation, and the solutions of this functional equation are the *period functions*.

The functional equation can be deduced from the dynamics of the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$: A well-chosen discretization of the geodesic flow on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ gives rise to a discrete dynamical system on $(0,\infty)$, more precisely to a finitely-branched self-map on $(0,\infty)\smallsetminus\mathbb{Q}$, which is closely related to the Farey map. The associated transfer operator with parameter s, called *slow transfer operator*, is finite-term. The defining equation of its eigenfunctions with eigenvalue 1 is just the function equation from above.

Furthermore, an induction on parabolic elements in the discretization gives rise to a companion discrete dynamical system, an infinitely-branched self-map on $(0,\infty)$, closely related to the Gauss map and continued fraction expansions. The family of *fast transfer operators* associated to this map represents the Selberg zeta function via its Fredholm determinant and can be used to characterize necessary decay properties of period functions.

Turning around the order of this presentation (as it is done for generalizations), the geodesic flow by means of discretizations and transfer operator techniques gives rise to a functional equation suitable for the notion of period functions. And indeed the regularity and decay properties to be requested from period functions as well as the construction of the cohomology theory can partly be motivated by geometric-dynamical considerations. We refer to [32] and [9, Section 8] for more explanations.

These results have been generalized to certain classes of hyperbolic orbisurfaces of finite and infinite area. See in particular [9, 10, 26, 28–31]. With this paper we reach out to establish first instances of analogous results beyond Maass forms of weight 0 as well as beyond hyperbolic orbisurfaces. We provide such results for

(a) Jacobi Maass cusp forms of any real weight for the discrete (integral) Jacobi group of level 1, which is the semi-direct product of $\mathrm{SL}_2(\mathbb{Z})$ and the integer points $\mathsf{Hei}(\mathbb{Z})$ of the Heisenberg group, and
(b) vector-valued Maass cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ of any real weight and any unitary representation.

To survey our results in more detail, we start with a few preparatory comments and explanations. We set throughout $G := \mathrm{SL}_2(\mathbb{R})$ and $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. We let Hei denote the 3-dimensional continuous Heisenberg group and $\mathsf{Hei}(\mathbb{Z})$ the discrete Heisenberg group, i.e., the subgroup of Hei given by restricting to the ring of integers. See Section 8 for precise definitions.

The space on which Jacobi Maass forms (and Jacobi Maass cusp forms) are defined is the product space $\mathfrak{H}\times\mathbb{C}$ of the hyperbolic plane $\mathfrak{H}$ and the complex plane $\mathbb{C}$. On this space, the (continuous) Jacobi group $G^J := \mathsf{Hei}\rtimes G$ (of level 1) acts by fractional linear transformations in the Hei-component and by a certain skew product in the $\mathbb{C}$-component. See Section 8. Endowing $\mathfrak{H}\times\mathbb{C}$ with a Riemannian metric such that G^J acts by Riemannian isometries is not unique. Indeed, there is at least a two-parameter family of such Riemannian metrics on $\mathfrak{H}\times\mathbb{C}$ (see [36, Remark 2.5]). Thus, if we wanted to proceed as for hyperbolic orbisurfaces starting with a discretization of the geodesic flow, then we would face the difficulty of the non-uniqueness of the choice of this flow. In addition, even if we settled on one choice of the Riemannian metric, then we would need to handle the seven-dimensional sphere bundle of $\mathfrak{H} \times \mathbb{C}$ in combination with a mixture of hyperbolic and euclidean action behavior.

To circumvent this obstacle and to simultaneous stay close to the approach for hyperbolic orbisurfaces, we use here another approach based on the theta decomposition. Pitale [27, Theorem 4.6] showed that Jacobi Maass forms for $\Gamma^J := \mathsf{Hei}(\mathbb{Z}) \rtimes \Gamma$ of integral weight and positive integral index are linear isomorphic to certain spaces of vector-valued Maass forms on $\Gamma\backslash\mathfrak{H}$. Thus, this isomorphism allows us to transfer the quest for a notion of period functions for Jacobi Maass cusp forms to a question about period functions for vector-valued Maass cusp forms in the more well-known realm of hyperbolic surfaces. This way, the request for a discretization of the non-unique geodesic flow on $\Gamma^J\backslash(\mathfrak{H} \times \mathbb{C})$ is solved implicitly and essentially avoided. However, via this isomorphism the *integral weight* of Jacobi Maass forms gets converted into a *half-integral weight* for the vector-valued Maass forms. Up to date, only *weight-zero situations* have been considered in the literature in this realm of research, and hence we are required to find a notion of period functions for Maass cusp form for half-integral weight and establish the necessary linear isomorphism. Indeed, we provide these results for *arbitrary real weight* as it is no more difficult than half-integral weights. In turn, the generality of our results for vector-valued Maass cusp forms then allows us to consider also arbitrary real weight for Jacobi Maass cusp forms.

For the definition of vector-valued Maass forms of weight $k \in \mathbb{R}$ we fix a one-dimensional multiplier system v_k, given in (2.6), and a unitary representation ρ of Γ on a finite-dimensional vector space X_ρ. *Maass forms* of weight k, spectral parameter s and multiplier system ρv_k are smooth eigenfunctions $\mathfrak{H} \to X_\rho$ with eigenvalue $s(1-s)$ of the generalized Laplacian

$$\Delta_k := -y^2\,\partial_x^2 - y^2\,\partial_y^2 + iky \qquad (z = x + iy \in \mathfrak{H})$$

with growing at most polynomial towards ∞ and being invariant under the action $|_{\rho v_k,k}$ on all of Γ, where

$$u|_{\rho v_k,k}\gamma(z) := \rho(\gamma)^{-1} v_k(\gamma)^{-1} e^{-ik\arg(cz+d)} u(\gamma z)$$

for $u\colon \mathfrak{H} \to X_\rho$, $\gamma \in \Gamma$, $z \in \mathfrak{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The space of such Maass cusp forms is denoted $\mathcal{A}_k(s, \rho v_k)$. Asking for exponential decay towards ∞ instead of polynomial bounds defines the space of *Maass cusp forms* of weight k, spectral parameter s and multiplier system ρv_k, which is denoted $\mathcal{A}_k^0(s, \rho v_k)$. See Section 2 for more details. This section also contains an alternative definition using the universal covering group of G, which is helpful for our considerations.

The space $\mathsf{FE}^{\omega}_{\rho v_k,s,k}$ of period functions for weight k, spectral parameter s and multiplier system ρv_k consists of real-analytic functions $f\colon (0,\infty) \to X_\rho$ that obey certain extension properties and satisfy the three-term functional equation

$$f = f|^{\mathrm{ps}}_{\rho v_k,s,k}T + f|^{\mathrm{ps}}_{\rho v_k,s,k}T' \qquad \text{with } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } T' := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where the action $|^{\mathrm{ps}}_{\rho v_k,s,k}$ is closely related to the action $|_{\rho v_k,k}$. We refer to Section 3 for precise definitions. We emphasize that this functional equation can be deduced from a transfer operator associated to a discretization of the geodesic flow on the modular surface $\Gamma\backslash\mathfrak{H}$, and hence the same discretization as for the weight-zero results is lurking in our considerations.

Theorem A. *For $k \in \mathbb{R}$ and $s \in \mathbb{C}$ such that $\operatorname{Re} s \in (0,1)$ and $s \not\equiv \pm k/2 \bmod 1$, the vector spaces $\mathcal{A}^0_k(s,\rho v_k)$ and $\mathsf{FE}^{\omega}_{\rho v_k,s,k}$ are isomorphic.*

The isomorphism in Theorem A is indeed constructive, at least in the direction $\mathcal{A}^0_k(s,\rho v_k) \to \mathsf{FE}^{\omega}_{\rho v_k,s,k}$. It is given by an integral transform and uses a cohomological setting. The condition $s \not\equiv \pm k/2 \bmod 1$ in Theorem A restricts the spectral parameter s to the values for which the discrete series representation is irreducible. We refer to the full statement of this isomorphism in Theorem 7.1, Proposition 4.2 and their proofs.

In our present consideration we restrict to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ for definiteness and simplification of some steps. In particular, we may work with the Farey tesselation of $\mathfrak{H}$, which is underlying both the transfer-operator-based deduction of the functional equation above as well as some parts in the cohomological argumentations. However, we expect that the tools we use here for the generalization of the weight-zero results to arbitrary real weights can be adapted to non-cusp forms and other discrete subgroups of G.

Jacobi Maass forms for Γ^J of index $m \in \mathbb{Z}$, weight $k \in \mathbb{R}$, eigenvalue parameter $s \in \mathbb{C}$ and multiplier system $\varphi_a v_k$ with φ_a being a character parameterized by $a \in \mathbb{Z} \bmod 12$, are smooth functions $\mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ that are eigenfunctions of certain differential operators related to the Laplacian on $\mathfrak{H}$, that are of at most polynomial growth, and that are invariant under the action $|^J_{\varphi_a v_k,k,m}$, which is an extension of the action on Maass forms to include the elliptic variable space $\mathbb{C}$. We refer to Section 8 for details and an alternative definition using the universal covering group. The space of such Jacobi Maass forms is denoted $\mathcal{A}^J_{k,m}(s,\varphi_a\chi_k)$. Asking for quick decay instead of polynomial growth, defines the subspace of *Jacobi Maass cusp forms*, denoted $\mathcal{A}^{J,0}_{k,m}(s,\varphi_a\chi_k)$. We obtain the following generalization of Pitale's result, where $\rho_{a,m}$ is a certain unitary representation of Γ which is associated to φ_a and m, defined in (8.36).

Theorem B. *Let $m \in \mathbb{Z}$, $m \geq 1$, $a \in \mathbb{Z}/12$, $k \in \mathbb{R}$ and $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$. Set $s' := (s+1)/2$ and $k' := k - 1/2$. Then the vector spaces $\mathcal{A}^J_{k,m}(s,\varphi_a\chi_k)$ and $\mathcal{A}_{k'}(s',\rho_{a,m}v_{k'})$ are isomorphic, as well as the vector spaces $\mathcal{A}^{J,0}_{k,m}(s,\varphi_a\chi_k)$ and $\mathcal{A}^0_{k'}(s',\rho_{a,m}v_{k'})$.*

Theorem B provides a theta decomposition, which generalizes Pitale's result. It is based on working with the universal covering group of G^J and Fourier expansions. Indeed, the proof provides more insights into the isomorphism. We refer to Theorem 8.3 and its proof for full details.

The combination of (the full versions of) Theorem A and Theorem B yields period functions for Jacobi Maass cusp forms.

Theorem C. *Let $m \in \mathbb{Z}$, $m \geq 1$, $s \in \mathbb{C}$ and $k \in \mathbb{R}$ such that* $\operatorname{Re} s \in [0, 1)$ *and* $s \not\equiv \pm k \bmod 2$. *Set $s' := (s + 1)/2$ and $k' := k - 1/2$. Then the vector spaces $\mathcal{A}^{J,0}_{k,m}(s, \varphi_a\chi_k)$ and $\mathsf{FE}^{\omega}_{\rho_{a,m}v_{k'},s',k'}$ are isomorphic.*

This result is stated as Corollary 8.4 in Section 8. As for Theorem A, the isomorphism between Jacobi Maass cusp forms and their period functions can be provided rather explicitly via an integral transform. We refer to Proposition 8.5 for details.

This article is structured as follows. In Section 2 we introduce Maass forms and Maass cusp forms for arbitrary real weight, first as functions on $\mathfrak{H}$ and then as functions on the universal covering group of G. We discuss their Fourier expansions, and weight-increasing and weight-lowering between Maass forms, which yields that only spectral parameters s with $\operatorname{Re} s \in (0, 1)$ need to be considered (Proposition 2.5). In Section 3 we discuss principal series representations and discrete series representations in the presence of arbitrary real weight. We further provide the definition of period functions and show some first properties. In Section 4 we intensify the discussion of period functions, provide the integral transform including the generalization of all necessary ingredients, present the cohomology setting, in particular the parabolic cocycles, and establish an explicit linear map from Maass cusp forms to period functions (Proposition 4.2). In Section 5 we discuss the relation between slow/fast transfer operators and period functions. We obtain that, as in the classical results, slow transfer operators determine the functional equation (and some parts of the regularity conditions) of period functions, and fast transfer operators help to characterize the necessary regularity conditions. In Section 6 we start working on showing that the linear map from Maass cusp forms to period functions is indeed bijective by indeed inverting this map, i.e., the integral transform. To that end we provide a kernel function for the inversion, and a boundary germ construction. In Section 7 we complete these efforts by constructing the inverse map from period functions to Maass cusp forms. In Section 8 we provide a generalization of Jacobi Maass forms to arbitrary real weight, we establish a theta decomposition for them, allowing us to relate Jacobi Maass forms and vector-valued Maass forms for real weight, and we apply our result on period functions for Maass cusp forms to obtain period functions for Jacobi Maass cusp forms. Throughout we attempt to follow the proofs in the previous results mentioned at the beginning of this introduction as close as possible, and we emphasize the new tools and steps needed for the generalizations.

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2. Maass forms

We consider in Subsection 2.1 Maass forms first as functions on the complex upper half-plane that are eigenfunctions of a generalized Laplace operator, satisfy an invariance relation and a growth condition, and we discuss their Fourier expansion in Subsection 2.2.

For our purposes it is useful to consider Maass forms also as functions on a Lie group covering $\mathrm{SL}_2(\mathbb{R})$. We discuss this in Subsection 2.3. Further, the action of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$, in Subsection 2.4, can be used to relate Maass forms in weights that differ by a multiple of 2. This leads to Proposition 2.5, which reduces the set of eigenvalues of the Laplace operator that we have to consider.

2.1. Maass forms on the upper half-plane.

We first discuss the concepts involved in the definition of Maass forms, working with functions on the complex upper half-plane $\mathfrak{H} = \{z \in \mathbb{C} \ : \ \mathrm{Im}\, z > 0\}$.

Differential equation. Maass forms of weight $k \in \mathbb{R}$ should be eigenfunctions of the differential operator

$$\Delta_k \;=\; -y^2\,\partial_x^2 - y^2\,\partial_y^2 + iky\,\partial_x\,. \tag{2.1}$$

Here and further on we will tacitly write $z \in \mathfrak{H}$ as $z = x + iy$ with $x \in \mathbb{R}$, $y > 0$. For $k = 0$, the differential operator Δ_0 is the hyperbolic Laplace operator on $\mathfrak{H}$. For any $k \in \mathbb{R}$ the operator Δ_k is elliptic, and all its eigenfunctions are real-analytic. The operator Δ_k makes sense on vector-valued functions by applying it on each coordinate component. We follow the practice of parametrizing eigenvalues as $s(1-s)$ with $s \in \mathbb{C}$, and call s and $1-s$ *spectral parameters*. Maass forms satisfy the condition

$$\Delta_k u \;=\; s(1-s)u \tag{2.2}$$

for some $s \in \mathbb{C}$.

Invariance under the modular group. For each $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}) =: G$ we use the operator

$$u \mapsto u|_k g\,, \quad (u|_k g)(z) \;=\; e^{-ik\arg(cz+d)}\,u(gz)\,, \tag{2.3}$$

where we take $-\pi < \arg(cz+d) \le \pi$. By gz we mean $\frac{az+b}{cz+d}$. The operators can be applied to vector-valued functions. The operators $|_k g$ commute with the operator Δ_k.

For $k \in \mathbb{R} \smallsetminus \mathbb{Z}$ the operator $|_k g$ depends on the choice of the argument. This has the consequence that for $g = \mathrm{k}(\vartheta) := \left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right)$ and $z = i$ the factor $e^{-ik\arg(\cos\vartheta - i\sin\vartheta)}$ is right continuous in $\vartheta = -\pi$, but not continuous if $k \notin 2\mathbb{Z}$.

If $k \in \mathbb{Z}$, then $g \mapsto |_k g$ is a (right) representation of G in the functions on $\mathfrak{H}$:

$$u|_k g_1 g_2 \;=\; (u|_k g_1)\,\big|_k\, g_2 \qquad (g_1, g_2 \in G)\,. \tag{2.4}$$

For $k \in \mathbb{R} \smallsetminus \mathbb{Z}$ this relation holds only up to a factor with absolute value 1. The operators $|_k g$ with $g \in G = \mathrm{SL}_2(\mathbb{R})$ generate a group, which depends on k. This group is a homomorphic image of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, which we will discuss in Subsection 2.3.

It is impossible to add a factor in the definition in (2.3) such that we arrive at a representation of the group G. However we can turn $\gamma \mapsto |_k\gamma$ into a representation of the discrete subgroup $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ by writing

$$u|_{v_k,k}\gamma(z) \;=\; v_k(\gamma)^{-1}\, e^{-ik\arg(cz+d)}\, u(\gamma z)\,, \qquad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.5}$$

and the function $v_k\colon \Gamma \to \mathbb{C}^*$ given by

$$\begin{aligned} v_k(\gamma) \;&=\; \frac{\eta^{2k}(\gamma z)}{(c_\gamma z + d_\gamma)^k\, \eta^{2k}(z)}\,, \\ \eta^{2k}(z) \;&=\; e^{\pi i k z/6} \prod_{n\ge 1} \Bigl(1 - e^{2\pi i n z}\Bigr)^{2k}\,, \end{aligned} \tag{2.6}$$

where η is the Dedekind eta function. For the multiplier system v_k, we have in fact an action of $\mathrm{PSL}_2(\mathbb{Z}) = \Gamma/\{\pm I_2\}$ with $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, since $|_{v_k,k}\gamma = |_{v_k,k}(-\gamma)$ for all $\gamma \in \Gamma$.

We let $X_\rho = \mathbb{C}^n$ for some $n = n(\rho) \in \mathbb{Z}_{\ge 1}$, and consider a unitary representation $\rho\colon \Gamma \to \mathrm{U}(X_\rho)$ with respect to the standard inner product $(x, y)_\rho = \sum_l x_l \bar{y}_l$ of $\mathbb{C}^n = X_\rho$. We obtain a representation on the X_ρ-valued functions on $\mathfrak{H}$ by

$$u|_{\rho v_k,k}\gamma \;=\; \rho(\gamma)^{-1}\, u|_{v_k,k}\gamma\,. \tag{2.7}$$

When dealing with the Jacobi group we will obtain examples of representations with these properties.

Growth conditions. On functions u that are invariant under $|_{\rho v_k,k}\Gamma$ we impose growth conditions at ∞. A function u has *polynomial growth* if

$$u(z) \;=\; \mathrm{O}(y^a) \qquad \text{as } y \uparrow \infty \tag{2.8}$$

for some $a \in \mathbb{R}$ that may depend on u, uniform for x in compact sets in $\mathbb{R}$. A function u has *exponential decay* if for some $a > 0$

$$u(z) \;=\; \mathrm{O}(e^{-ay}) \qquad \text{as } y \uparrow \infty, \text{ uniformly for } x \text{ in compact sets in } \mathbb{R}\,. \tag{2.9}$$

Definition 2.1. Let $s \in \mathbb{C}$, $k \in \mathbb{R}$, and $\rho\colon \Gamma \to \mathrm{U}(X_\rho)$ a finite-dimensional unitary representation of Γ. The space $\mathcal{A}_k(s, \rho v_k)$ of *Maass forms* of weight k, with spectral parameter s and multiplier system ρv_k consists of all smooth functions $u\colon \mathfrak{H} \to X_\rho$ that satisfy $u|_{\rho v_k,k}\gamma = u$ for all $\gamma \in \Gamma$, the eigenfunction condition (2.2), and the condition (2.8) of polynomial growth. The stronger condition (2.9) of exponential decay determines the subspace $\mathcal{A}_k^0(s, \rho v_k)$ of *Maass cusp forms*.

The presence of $-I_2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in Γ requires attention. We have

$$v_k(-I_2) e^{-ik\arg(-1)} \;=\; 1 \tag{2.10}$$

from (2.6). Hence we get $\rho(-I_2)u(z) = u(z)$. So functions that are invariant under $|_{\rho v_k,k}\Gamma$ have values in the 1-eigenspace of $\rho(-I_2)$. We might avoid this by requiring that ρ is a representation of $\mathrm{PSL}_2(\mathbb{Z})$. However, the representation ρ might arise naturally, and it might be inconvenient to tamper with it.

The concept of Maass form in the scalar-valued case for more general groups than $\mathrm{SL}_2(\mathbb{Z})$ is due to H. Maass, who called them *non-analytic modular forms*; see

[21] and [22, p. 185]. For vector-valued Maass forms we may consult Roelcke [33, 34].

2.2. **Fourier expansion.** Let the function $u\colon \mathfrak{H} \to X_\rho$ be equivariant under $|_{\rho v_k,k}T$ with $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. The operator $v_k(T)\rho(T)$ is unitary, and X_ρ has an orthonormal basis of eigenvectors $\{e_l : 1 \le l \le n(\rho)\}$ of $v_k(T)\rho(T)$. We take $\kappa_l \in [0,1)$ such that $\rho(T)v_k(T)e_l = e^{2\pi i\kappa_l}\, e_l$. Writing $u\colon \mathfrak{H} \to X_\rho$ in the form

$$u(z) \;=\; \sum_l u_l(z)\, e_l\,, \tag{2.11}$$

we get $n(\rho)$ component functions $u_l\colon \mathfrak{H} \to \mathbb{C}$.

If $u \in \mathcal{A}^0_k(s,\rho v_k)$, then the functions u_l have a Fourier expansion

$$u_l(z) \;=\; \sum_{\substack{n\equiv\kappa_l \bmod 1,\\ n\neq 0}} c_l(n) e^{2\pi i n x}\, W_{\varepsilon(n)k/2,s-1/2}(4\pi|n|y)\,, \tag{2.12}$$

with $\varepsilon(n) = \mathrm{Sign}(n)$. We note that n runs through a set of real numbers, not necessarily integers. Since the W-Whittaker functions and their derivatives have exponential decay, all derivatives of u with respect to x and y satisfy condition (2.9) of exponential decay. Under less strict assumptions than the unitarity of ρ there still is a Fourier expansion in which W-Whittaker functions are involved; see [13]. The exponential decay of derivatives of Maass cusp forms goes through.

The Fourier terms of components u_l of a function $u\colon \mathfrak{H} \to X_\rho$ satisfying only (2.2) and (2.7) are more general. For $\mathrm{Re}\, s > 0$ any term with order $n \neq 0$ is a linear combination of

$$e^{2\pi i n x}\, W_{\varepsilon(n)k/2,s-1/2}(4\pi|n|y) \quad\text{and}\quad e^{2\pi i n x}\, M_{\varepsilon(n)k/2,s-1/2}(4\pi|n|y)\,. \tag{2.13}$$

The W-Whittaker function has exponential decay, and the M-Whittaker function has exponential growth. The term of order 0 is for $s \neq \frac12$ a linear combination of y^s and y^{1-s}, and for $s = \frac12$ a linear combination of $y^{1/2}$ and $y^{1/2}\log y$.

Fourier terms inherit growth conditions. A consequence is that if we replace the condition of exponential growth of Maass cusp forms by $u(z) \;=\; \mathrm{O}(y^{-a})$ as $y \uparrow \infty$ with $a > \max(\mathrm{Re}\, s, 1-\mathrm{Re}\, s)$, then we have Fourier expansions of the u_l as indicated in (2.12). This is the condition of *quick decay*. It is weaker than exponential decay, but for Maass forms quick decay implies exponential decay.

2.3. **Universal covering group.** The weight k of a Maass form as defined in Definition 2.1 is a parameter in the transformation behavior by elements of Γ. The concept of Maass forms on a Lie group separates the weight and the Γ-invariance. In the context of arbitrary real weights the Lie group to be used is the universal covering group of $\mathrm{SL}_2(\mathbb{R})$.

Description of the universal covering group. The Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$ writes each element of $\mathrm{SL}_2(\mathbb{R})$ uniquely as $\mathrm{p}(z)\mathrm{k}(\vartheta)$, $z \in \mathfrak{H}$ and $\vartheta \in \mathbb{R} \bmod 2\pi\mathbb{Z}$, with

$$\text{(2.14)}\qquad \begin{aligned} \mathrm{p}(x+iy) &= \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \qquad x+iy \in \mathfrak{H}\,, \\ \mathrm{k}(\vartheta) &= \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \quad \vartheta \in \mathbb{R}\,. \end{aligned}$$

The universal covering group $\tilde{G}$ of $G = \mathrm{SL}_2(\mathbb{R})$ is based on the covering

$$\mathfrak{H} \times \mathbb{R} \to \mathfrak{H} \times \mathbb{R}/2\pi\mathbb{Z}\,,$$

with the natural map $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. There exists a unique Lie group $\tilde{G}$ consisting of the elements $\tilde{\mathrm{p}}(z)\tilde{\mathrm{k}}(\vartheta)$ with $z \in \mathfrak{H}$ and $\vartheta \in \mathbb{R}$ with a group structure such that

$$\text{(2.15)}\qquad \mathrm{pr} : \tilde{\mathrm{p}}(z)\tilde{\mathrm{k}}(\vartheta) \mapsto \mathrm{p}(z)\mathrm{k}(\vartheta)$$

is a surjective Lie group homomorphism $\mathrm{pr} : \tilde{G} \to G$ with kernel

$$\text{(2.16)}\qquad \tilde{Z}_2 = \{\tilde{\mathrm{k}}(2\pi n) \;:\; n \in \mathbb{Z}\}\,.$$

One finds a description of the group operations in [3, §2.2.1].

The map pr in (2.15) is a group homomorphism. There does not exist an inverse group homomorphism, but we can choose a section $g \mapsto \ell(g)$ from G to $\tilde{G}$ of the homomorphism $\mathrm{pr} : \tilde{G} \to G$ by

$$\text{(2.17)}\qquad \ell\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \tilde{\mathrm{p}}\Big(\frac{ai+b}{ci+d}\Big)\tilde{\mathrm{k}}(-\arg(ci+d)).$$

We stress that the map $\mathrm{pr} \circ \ell$ is the identity on G, but pr is not invertible. Further for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ we have

$$\text{(2.18)}\qquad \ell\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tilde{\mathrm{p}}(z)\tilde{\mathrm{k}}(\vartheta) = \tilde{\mathrm{p}}\Big(\frac{az+b}{cz+d}\Big)\tilde{\mathrm{k}}(\vartheta - \arg(cz+d))\,,$$

where we use the argument convention $-\pi < \arg(cz+d) \le \pi$.

Weights and equivariance. A function f on $\tilde{G}$ has *weight* $k \in \mathbb{R}$ if it satisfies

$$\text{(2.19)}\qquad f(\tilde{g}\tilde{\mathrm{k}}(\vartheta)) = e^{ik\vartheta} f(\tilde{g}) \qquad \text{for all } \tilde{g} \in \tilde{G},\ \vartheta \in \mathbb{R}\,.$$

The representation of $\tilde{G}$ by right translation in the functions on $\tilde{G}$ is defined by

$$\text{(2.20)}\qquad (R(\tilde{g}_1)f)(\tilde{g}) = f(\tilde{g}\tilde{g}_1)\,.$$

Hence the function f on $\tilde{G}$ has weight k if the subgroup $\tilde{K} = \{\tilde{\mathrm{k}}(\vartheta) \;:\; \vartheta \in \mathbb{R}\}$ of $\tilde{G}$ acts according to the character $\tilde{\mathrm{k}}(\vartheta) \mapsto e^{ik\vartheta}$ of $\tilde{K}$. The representation of $\tilde{G}$ by left translation of function on $\tilde{G}$ is given by

$$\text{(2.21)}\qquad (L(\tilde{g}_1)f)(\tilde{g}) = f(\tilde{g}_1\tilde{g})\,.$$

As we do not apply $\tilde{g}_1^{-1}$ in (2.21), this is a right representation:

$$L(\tilde{g}_1\tilde{g}_2) = L(\tilde{g}_2)L(\tilde{g}_1)\,.$$

Since left and right translations commute, left translation does not change the weight of functions on $\tilde{G}$.

Discrete subgroup. The discrete subgroup $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \subset G$ has an inverse image $\tilde\Gamma = \mathrm{pr}^{-1}\Gamma$ in $\tilde G$. The group $\tilde\Gamma$ is discrete in $\tilde G$, and it contains the center

$$\tilde Z = \{\tilde{\mathrm{k}}(\pi n) \;:\; n \in \mathbb{Z}\} \tag{2.22}$$

of $\tilde G$. The group $\tilde\Gamma$ is equal to $\tilde Z\,\ell(\Gamma)$. The group $\tilde\Gamma$ is generated by

$$\tilde T = \tilde{\mathrm{p}}(i+1) \quad \text{and} \quad \tilde S = \tilde{\mathrm{k}}(-\pi/2)\,, \tag{2.23}$$

which implies that

$$\begin{aligned} \tilde T &= \ell(T) \qquad \text{with} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\\ \tilde S &= \ell(S) \qquad \text{with} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.24}$$

The relations for $\tilde\Gamma$ are determined by

$$\tilde S^2 \tilde T = \tilde T \tilde S^2 \quad \text{and } (\tilde T \tilde S)^3 = \tilde S^2\,. \tag{2.25}$$

Suppose that a function $f \in C^\infty(\tilde G)$ is left-invariant under $\tilde\Gamma$ and has weight k. Since $\tilde S^2$ is central in $\tilde G$ we have

$$f(\tilde g) = f(\tilde S^2 \tilde g) = f(\tilde g \tilde S^2) = e^{-\pi i k} f(\tilde g)\,.$$

Hence we have $f = 0$ if $k \in \mathbb{R} \smallsetminus \mathbb{Z}$. Thus, for general real weights, functions on $\tilde G$ cannot be left invariant under $\tilde\Gamma$, and hence we have to be content to work with functions that are left equivariant for a suitable character of $\tilde\Gamma$, for instance for the character χ_k determined by

$$\chi_k(\tilde T) = e^{\pi i k/6}\,, \qquad \chi_k(\tilde S) = e^{-\pi i k/2}\,. \tag{2.26}$$

Then we deal with χ_k-equivariant functions of weight k that satisfy

$$f(\tilde\gamma \tilde g) = \chi_k(\tilde\gamma)\, f(\tilde g) \qquad \tilde\gamma \in \tilde\Gamma,\ \tilde g \in \tilde G\,. \tag{2.27}$$

Functions on $\mathfrak{H}$ and on $\tilde G$. For any $u \in C^\infty(\mathfrak{H})$ and any weight $k \in \mathbb{R}$ we define $\Psi_k u \in C^\infty(\tilde G)$ by

$$(\Psi_k u)(\tilde{\mathrm{p}}(z)\tilde{\mathrm{k}}(\vartheta)) = e^{ik\vartheta}\, u(z)\,. \tag{2.28}$$

The function $\Psi_k u$ has weight k. Moreover, the operator $|_k g$ in (2.3) corresponds to left translation on $\tilde G$:

$$\Psi_k(u|_k g) = L(\ell(g))\Psi_k u \qquad \text{for all } g \in G\,. \tag{2.29}$$

The operator Ψ_k works for vector-valued functions as well as for scalar-valued functions. We have $v_k(\gamma) = \chi_k(\ell(\gamma))$ for all $\gamma \in \Gamma$. If ρ is a representation of Γ in $\mathbb{C}^n$, then we define $\rho(\tilde\gamma) := \rho(\mathrm{pr}\,\tilde\gamma)$ for all $\tilde\gamma \in \tilde\Gamma$. With (2.7) we get

$$\Psi_k(u|_{\rho v_k,k}\gamma) = \rho(\gamma)^{-1} L(\gamma) \Psi_k u\,. \tag{2.30}$$

2.4. **Lie algebra and differential operators.** The groups G and $\tilde{G}$ have isomorphic neighborhoods of the unit element, and hence they have the same Lie algebra $\mathfrak{g}$, with complexification $\mathfrak{g}_c = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$. We use the notations of Berndt and Schmidt, in particular for the basis $\{Z, X_+, X_-\}$ of the complexified Lie algebra. See [2, p. 12].

The Lie algebra $\mathfrak{g}_c$ acts on $C^\infty(\tilde{G})$ by differentiation on the right. This action commutes with left translation. This holds in particular for the following operators, in the coordinates given by $(x, y, \vartheta) \leftrightarrow \tilde{\mathrm{p}}(x+iy)\tilde{\mathrm{k}}(\vartheta)$.

$$\begin{aligned} Z &= -i\partial_\vartheta\,, \\ X_+ &= e^{2i\vartheta}\Big(2iy\,\partial_z - \frac{i}{2}\,\partial_\vartheta\Big) &&= e^{2i\vartheta}\Big(iy\,\partial_x + y\,\partial_y - \frac{i}{2}\,\partial_\vartheta\Big) \\ X_- &= e^{-2i\vartheta}\Big(-2iy\,\partial_{\bar{z}} + \frac{i}{2}\partial_\vartheta\Big) &&= e^{-2i\vartheta}\Big(-iy\,\partial_x + y\,\partial_y + \frac{i}{2}\,\partial_\vartheta\Big). \end{aligned} \tag{2.31}$$

The differential operator Z detects the weight of functions. With the commutator relations in $\mathfrak{g}_c$ (see [2, p. 12]) we check that X_+ shifts the weight up by 2, and X_- shifts down by 2. We have the second order element

$$\Delta = -X_-X_+ - \frac{1}{4}Z^2 - \frac{1}{2}Z = -X_+X_- - \frac{1}{4}Z^2 + \frac{1}{2}Z\,, \tag{2.32}$$

which is known to commute with all elements of $\mathfrak{g}_c$. The operator Δ is called the Casimir operator; it corresponds to the differential operator

$$-y^2(\partial_x^2 + \partial_y^2) + y\,\partial_x\partial_\vartheta\,. \tag{2.33}$$

On functions of weight k the operator Δ acts as

$$-y^2(\partial_x^2 + \partial_y^2) + iky\,\partial_x\,, \tag{2.34}$$

which is the operator Δ_k in (2.1).

Now we are ready to use the operator Ψ_k to transform Definition (2.1) into an equivalent definition of Maass forms as functions on $\tilde{G}$. We put

$$\tilde{\mathrm{a}}(y) := \tilde{\mathrm{p}}(iy) = \ell\begin{bmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{bmatrix}.$$

Definition 2.2. Let $s \in \mathbb{C}$, $k \in \mathbb{R}$, and $\rho\colon \Gamma \to \mathrm{U}(X_\rho)$ a finite-dimensional unitary representation of Γ. The space $\mathcal{A}_k(s, \rho\chi_k)$ of *Maass forms* on $\tilde{G}$ of weight k, with spectral parameter s and representation $\chi_k\rho$ consists of all smooth functions

$$f\colon \tilde{G} \to X_\rho$$

that satisfy

(a) $R(\tilde{\mathrm{k}}(\vartheta))f = e^{ik\vartheta} f$ for all $\vartheta \in \mathbb{R}$,
(b) $L(\tilde{\gamma})f = \rho(\tilde{\gamma})^{-1}\chi_k(\tilde{\gamma})\,f$ for all $\tilde{\gamma} \in \tilde{\Gamma}$,
(c) $\Delta f = s(1-s)f$,
(d) $f(\tilde{\mathrm{a}}(t)\tilde{g}) = \mathrm{O}(t^a)$ as $t \uparrow \infty$ uniform for $\tilde{\gamma}$ in compact sets in $\tilde{G}$ for some $a \in \mathbb{R}$.

If we replace condition (d) by $f(\tilde{\mathrm{a}}(t)\tilde{g}) = \mathrm{O}(t^{-a})$ as $t \uparrow \infty$, uniform for $\tilde{\gamma}$ in compact sets in $\tilde{G}$ for all $a \in \mathbb{R}$, then f is a *Maass cusp form*.

Weight shifting operators. The operator Δ preserves the weight of functions on $\tilde{G}$, and corresponds to the operator Δ_k on $\mathfrak{H}$.

The operator X_+ transforms functions of weight k into functions of weight $k+2$. It corresponds to the operator $X_{+,k} = 2iy\,\partial_z + \frac{k}{2}$ on $\mathfrak{H}$, which we can see with the following computation:

$$\begin{aligned} X_+(\Psi_k u)(z,\vartheta) &= e^{2i\vartheta}\Big(2iy\,\partial_z - \frac{i}{2}\,\partial_\vartheta\Big)\Big(u(z)e^{ik\vartheta}\Big)\\ &= 2iy\,\frac{\partial u}{\partial z}(z)e^{i(k+2)\vartheta} + u(z)\frac{k}{2}e^{i(k+2)\vartheta}\\ &= \Big(2iy\frac{\partial u}{\partial_z}(z) + \frac{k}{2}\,u(z)\Big)\,e^{i(k+2)\vartheta}\\ &= \Big(\Psi_{k+2}\Big(2iy\partial_z + \frac{k}{2}\Big)u\Big)(z,\vartheta)\,. \end{aligned}$$

We leave to the reader the analogous computation for X_- to see that X_- corresponds to $X_{-,k} = -2iy\,\partial_{\bar{z}} - \frac{k}{2}$. The operators $X_{+,k}$ and $-X_{-,k}$ correspond to the operators in [33, (3.1), (3.2)]. Working with Maass forms on the group we have the following results.

Lemma 2.3. *The weight shifting operators satisfy*

$$X_\pm : \mathcal{A}^0_k(s,\chi_k\rho) \to \mathcal{A}^0_{k\pm 2}(s,\chi_k\rho)\,. \tag{2.35}$$

We note that the weight changes, but that the representation $\chi_k\rho$ of $\tilde{\Gamma}$ stays the same.

Proof. We apply the operator $X_{\pm,k}$ corresponding to $X_\pm$ to the Fourier expansion in (2.12). This leads to an expression in the Whittaker function and its derivative. The contiguous relations for Whittaker functions allow to show that we get a multiple of the expression for the Fourier expansion in weight $k \pm 2$. The absolute convergence of the Fourier expansion gives an estimate of the growth of the coefficients $c_l(n)$ that is strong enough to show convergence of the sum of derivatives, and it leads to exponential decay of the resulting sum. □

Lemma 2.4. *The operators $X_\pm X_\mp$ act in $\mathcal{A}^0_k(s,\chi_k\rho)$ as multiplication by the factor $(s\mp k/2)(s-1\pm k/2)$.*

Proof. The operator Z acts in $\mathcal{A}^0_{k'}(s,\chi_k\rho)$ as multiplication by k'. The operator Δ acts as multiplication by $s(1-s)$. A computation based on (2.32) gives the actions of $X_\pm X_\mp$. □

If $s \not\equiv \frac{k}{2}$ mod 1 and $s \not\equiv -\frac{k}{2}$ mod 1 the product $(s\mp k/2)(s-1\pm k/2)$ is non-zero, and hence

$$X_{\pm,k} : \mathcal{A}^0_k(s,\rho v_k) \to \mathcal{A}^0_{k\pm 2}(s,\rho v_k) \tag{2.36}$$

is a bijection. (Note that the multiplier system ρv_k is preserved under $X_{\pm,k}$.) This shows that if we know one space $\mathcal{A}^0_k(s,\rho v_k)$ with $s \not\equiv \pm\frac{k}{2}$ mod 1, then we know $\mathcal{A}^0_{k'}(s,\rho v_k)$ for all $k' \equiv k$ mod 2.

Proposition 2.5. *Let $s \in \mathbb{C}$, $s \not\equiv \pm\frac{k}{2} \bmod 1$. Define $\nu \in [0,2)$ by $\nu \equiv k \bmod 2$. If the space $\mathcal{A}^0_k(s, \rho v_k)$ is non-zero, then*

$$\begin{aligned} & s \in \frac{1}{2} + (i\mathbb{R} \smallsetminus \{0\}), \\ \text{or} \quad & \begin{cases} \frac{\nu}{2} < s < 1 - \frac{\nu}{2} & \text{if } \nu \in [0,1), \\ 1 - \frac{\nu}{2} < s < \frac{\nu}{2} & \text{if } \nu \in [1,2). \end{cases} \end{aligned} \tag{2.37}$$

This follows from [33, Satz 3.1], which implies that

$$-s(2-s) \;\leq\; -\frac{k'}{2}\Big(1 + \frac{k'}{2}\Big) \tag{2.38}$$

for all $k' \equiv k \bmod 2$. For $k' = \nu$ this equality is stronger than for all other $k' \equiv \nu \bmod 2$.

3. Principal series representation

In [7], the representation of G underlying the modules used in the cohomology of Γ is the discrete series representation. Handling arbitrary real weights requires some care.

Operators on the real projective line. The action of G on $\mathfrak{H}$ by fractional linear transformations extends to an action on the projective line $\mathbb{P}^1_{\mathbb{R}}$, which is the boundary of $\mathfrak{H}$.

By an open interval in $\mathbb{P}^1_{\mathbb{R}}$ we mean an open connected subset $I \subset \mathbb{P}^1_{\mathbb{R}}$ that is not equal to $\mathbb{P}^1_{\mathbb{R}}$ and has more than one point. The set $\mathbb{R}$ is an open interval in $\mathbb{P}^1_{\mathbb{R}}$, and intervals (α,β) in $\mathbb{R}$ with $\alpha < \beta \in \mathbb{R}$ are intervals in $\mathbb{P}^1_{\mathbb{R}}$ as well. If $\alpha > \beta$, then we have the open interval $(\alpha,\beta)_c = (\alpha,\infty) \cup \{\infty\} \cup (-\infty,\beta)$.

Let $s \in \mathbb{C}$ and $k \in \mathbb{R}$, and let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. For functions φ on an open subset $I \subset \mathbb{P}^1_{\mathbb{R}}$ we define $\varphi|^{\mathrm{ps}}_{s,k} g$ on $g^{-1}I$ by

$$\begin{aligned} (\varphi|^{\mathrm{ps}}_{s,k} g)(t) \;=\; & (a-ic)^{-s+k/2}\,(a+ic)^{-s-k/2}\Big(\frac{t-i}{t-g^{-1}i}\Big)^{s-k/2} \\ & \cdot \Big(\frac{t+i}{t-g^{-1}(-i)}\Big)^{s+k/2} \varphi(gt)\,, \end{aligned} \tag{3.1}$$

where we use $-\pi \leq \arg(a-ic) < \pi$ and $-\pi < \arg(a+ic) \leq \pi$. (In this way $\varphi|^{\mathrm{ps}}_{s,k}\mathrm{k}(\vartheta)$ is right-continuous in $\vartheta = -\pi$, like we have in (2.3).)

The function $t \mapsto \varphi(gt)$ is again a function on $\mathbb{P}^1_{\mathbb{R}}$. The function $t \mapsto \left(\frac{t-i}{t-g^{-1}i}\right)^{s-k/2}$ is well-defined on $\mathbb{P}^1_{\mathbb{R}}$. In fact it determines a holomorphic function on $\mathbb{P}^1_{\mathbb{C}}$ minus a path from i to $g^{-1}i$ in $\mathfrak{H}$. Similarly, the other factor is holomorphic on $\mathbb{P}^1_{\mathbb{C}}$ outside a path in the lower half-plane. These two factors are real-analytic on $\mathbb{P}^1_{\mathbb{R}}$. So $|^{\mathrm{ps}}_{s,k} g$ sends real-analytic functions on I to real-analytic functions. It also sends C^p-functions to C^p-functions for $p = 0, 1, \ldots, \infty$. Tensoring $\mathbb{C}$ with X_ρ we get operators

$$(\varphi|^{\mathrm{ps}}_{\rho v_k, s,k}\gamma)(t) \;=\; \rho(\gamma)^{-1}\, v_k(\gamma)^{-1}\,(\varphi|^{\mathrm{ps}}_{s,k}\gamma)(t) \tag{3.2}$$

for each $\gamma \in \Gamma$, and have

$$\varphi|^{\mathrm{ps}}_{\rho v_k,s,k}(\gamma_1\gamma_2) \;=\; \bigl(\varphi|^{\mathrm{ps}}_{\rho v_k,s,k}\gamma_1\bigr)|^{\mathrm{ps}}_{\rho v_k,s,k}\gamma_2 \tag{3.3}$$

for $\gamma_1, \gamma_2 \in \Gamma$.

In this way we arrive at the Γ-equivariant sheaf $\mathcal{V}^{\omega}_{\rho v_k,s,k}$ of analytic functions on $\mathbb{P}^1_{\mathbb{R}}$ with values in X_ρ, and on larger equivariant sheaves $\mathcal{V}^{p}_{\rho v_k,s,k}$ of p times continuously differentiable functions, with $p = 0, 1, 2, \ldots, \infty$. The action of $\gamma \in \Gamma$ is given by

$$|^{\mathrm{ps}}_{\rho v_k,s,k}\gamma \colon \mathcal{V}^{\omega}_{\rho v_k,s,k}(I) \to \mathcal{V}^{\omega}_{\rho v_k,s,k}(\gamma^{-1}I)\,. \tag{3.4}$$

The space of global sections $\mathcal{V}^{\omega}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ is Γ-invariant for this action. This is the *principal series representation* twisted by ρv_k. Similar remarks hold for the larger sheaves $\mathcal{V}^{p}_{\rho v_k,s,k}$ of p times continuously differentiable functions.

We note that $|^{\mathrm{ps}}_{s,k}(-I_2)$ is multiplication by $e^{-\pi i k}$, independent of s. From $v_k(-I_2) = e^{-\pi i k}$ we conclude that $|^{\mathrm{ps}}_{\rho v_k,s,k}(-I_2)$ is just application of the operator $\rho(-I_2)^{-1} = \rho(-I_2)$. (We use that ρ is a representation of Γ, and $-I_2 \in \Gamma$ is its own inverse.) For any set $I \subset \mathbb{P}^1_{\mathbb{R}}$ the 1-eigenspace of $|^{\mathrm{ps}}_{\rho v_k,s,k}(-I_2)$ in $\mathcal{V}^{\omega}_{\rho v_k,s,k}(I)$ is independent of s and k.

3.1. **Period functions.** The period functions that we want to relate to Maass cusp forms in $\mathcal{A}^0_k(s,\rho v_k)$ form a subspace of $\mathcal{V}^{\omega}_{\rho v_k,s,k}(0,\infty)$ with several additional properties.

Action of $-I_2$. Like for Maass cusps forms, we want period functions to have values in the 1-eigenspace of $\rho(-I_2)$. This implies that

$$(f|^{\mathrm{ps}}_{\rho v_k,s,k}S)|^{\mathrm{ps}}_{\rho v_k,s,k}S \;=\; f \tag{3.5}$$

for a period function f, for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Three term relation. We want the period functions to satisfy the three term relation

$$f \;=\; f|^{\mathrm{ps}}_{\rho v_k,s,k}T + f|^{\mathrm{ps}}_{\rho v_k,s,k}T'\,, \tag{3.6}$$

where $T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = ST^{-1}S$. This relation goes back to the three term relation in Lewis's paper [18].

For $f \in \mathcal{V}^{\omega}_{\rho v_k,s,k}(0,\infty)$, the terms on the right hand side of (3.6) are elements of $\mathcal{V}^{\omega}_{\rho v_k,s,k}(-1,\infty)$ and $\mathcal{V}^{\omega}_{\rho v_k,s,k}((0,-1)_c)$. In the relation these two terms are understood to be restricted to $(0,\infty)$.

Continuous extension. The period functions attached to Maass cusp forms by Lewis and Zagier [20] determine real-analytic functions on $(0,\infty)$ that satisfy the three term relation (3.6) with $\rho v_k = 1$, $\mathrm{Re}\, s \in (0,1)$, and $k = 0$, and have estimates at the boundary points of $(0,\infty)$. By [7, Theorem B and Proposition 14.2] these functions have, in the projective model used in this paper, a smooth extension f to $\mathbb{P}^1_{\mathbb{R}}$ satisfying $f|^{\mathrm{ps}}_{1,s,0}S = -f$. In particular, $\lim_{x\downarrow 0} f(x)$ and $\lim_{x\uparrow 0} f(x)$ exist and are equal, and analogously for the one-sided limits at ∞.

Here we require that the limits

$$a_\infty(f) = \lim_{t\uparrow\infty} f(t) \quad \text{and} \quad a_0(f) = \lim_{t\downarrow 0} f(t) \tag{3.7}$$

exist, and satisfy

$$a_0(f) = -\rho(S)a_\infty(f). \tag{3.8}$$

We note that $f|^{\mathrm{ps}}_{\rho v_k,s,k}S$ is defined on $(\infty,0)_c$. We have for $t<0$:

$$\begin{aligned}(f|^{\mathrm{ps}}_{\rho v_k,s,k}S)(t) &\overset{(3.1)}{=} \rho(S)^{-1}v_k(S)^{-1}(-i)^{k/2-s}\, i^{-s-k/2}\, f(-1/t)\\ &\overset{(2.6)}{=} \rho(S)^{-1}(-i)^{-k}\, e^{\pi i(-k)}\, f(-1/t)\\ &\overset{t\uparrow 0}{\longrightarrow} \rho(S)a_\infty(f)\,.\end{aligned}$$

So (3.8) ensures that the real-analytic function $-f|^{\mathrm{ps}}_{\rho v_k,s,k}S$ on $(\infty,0)_c$ has the same limit for $t\uparrow 0$ as the real-analytic function f on $(0,\infty)$ for $t\downarrow 0$. This implies that

$$t\mapsto \begin{cases} f(t) & \text{for } t>0\,,\\ -(f|^{\mathrm{ps}}_{\rho v_k,s,k}S)(t) & \text{for } t<0\,,\end{cases} \tag{3.9}$$

extends as a continuous function on $\mathbb{R}$.

Applying $|^{\mathrm{ps}}_{\rho v_k,s,k}S$ gives a similar continuous extension across ∞. One can check that the three term equation on $(0,\infty)$ implies that it also holds on $(\infty,-1)_c$ and on $(-1,0)$.

Holomorphic extension. A real-analytic function on $(0,\infty)$ is locally on $(0,\infty)$ given by power series, and hence extends holomorphically to a complex neighborhood of $(0,\infty)$. For period functions we require that the extension is possible to a wedge of the form

$$W_\delta = \{t\in\mathbb{C}\smallsetminus\{0\} \;:\; |\arg t|<\delta\}\,. \tag{3.10}$$

We note that the extensions of the three functions in the three term relation (3.6) may extend to different domains. The relation extends only to a connected neighborhood of $(0,\infty)$.

Definition 3.1. The space $\mathsf{FE}^{\omega}_{\rho v_k,s,k}$ of *period functions* is the linear space of elements $f\in\mathcal{V}^{\omega}_{\rho v_k,s,k}(0,\infty)$ that

(a) f has values in the 1-eigenspace of $|^{\mathrm{ps}}_{\rho v_k,s,k}(-I_2)$.
(b) f satisfies the three-term relation (3.6).
(c) f has limits at 0 and ∞ as indicated in (3.7) and (3.8).
(d) f has a holomorphic extension to a wedge W_δ for some $\delta\in(0,\pi/2)$.

The period functions in [7, p. 85] are characterized by a boundary condition, equivalent to O(1) at 0 and ∞ in the projective model used here. It is equivalent to the existence of asymptotic expansions at 0 and ∞. The existence of limits in part (c) is easier to handle, and leads to the same space of period functions.

Our aim is to establish a relation between Maass cusp forms in $\mathcal{A}^0_k(s,\rho v_k)$ and period functions in $\mathsf{FE}^{\omega}_{\rho v_k,s,k}$.

The properties required in parts (b) and (d) in Definition 3.1 allow us to apply the bootstrap method method in [20, Chap III.4, p. 240].

Proposition 3.2. *Each period function $f \in \mathsf{FE}^{\omega}_{\rho v_k,s,k}$ has a holomorphic extension to $\mathbb{C}' = \mathbb{C} \smallsetminus (-\infty, 0]$.*

Proof. Condition (d) implies that f is holomorphic on a wedge

$$W_\delta = \{z \in \mathbb{C}' \ : \ |\arg(z)| < \delta\}$$

for some small $\delta > 0$. For each $z \in \mathbb{C}'$ we can use the three term relation (3.6) to express $f(z)$ as a finite sum of translates $f|_{\rho v_k,s,k}\gamma(z)$ with γ in the semigroup generated by T and T' such that $f(\gamma z)$ is in W_δ. □

In the following result we use the orthonormal eigenbasis e_l of X_ρ, for the scalar product $(\cdot,\cdot)_\rho$, and the parameters $\kappa_l \in [0, 1)$ introduced in §2.2.

Lemma 3.3. *Let $f \in \mathsf{FE}^{\omega}_{\rho v_k,s,k}$. If $\kappa_l = 0$, then $\left(\rho(T')^{-1} v_k(T')^{-1} f(1), e_l\right)_\rho = 0$.*

Proof. We take the limit as $t \uparrow \infty$ of the three term equation, and project it to the line in X_ρ spanned by e_l:

$$\begin{aligned} a_\infty(f) \;&=\; \rho(T)^{-1} v_k(T)^{-1}\, 1^{s-k/2}\, 1^{-s-k/2}\, a_\infty(f) \\ &\qquad + \rho(T')^{-1} v_k(T')^{-1}\, (1-i)^{-s+k/2}\, (1+i)^{-s-k/2}\, 1^{s-k/2}\, 1^{s+k/2}\, f(1)\,, \\ (a_\infty(f), e_l)_\rho \;&=\; e^{-2\pi i\kappa_l}\, (a_\infty(f), e_l)_\rho + 2^{-s}\, i^{-k/2}\, (\rho(T')^{-1} v_k(T')^{-1} f(1), e_l)_\rho\,. \end{aligned}$$

For $\kappa_l = 0$ this gives the assertion in the lemma. □

We do not know how $\rho(T')$ acts on the eigenbasis for $\rho(T)$, and further simplification seems hard.

4. Period functions

In this section we show that we can associate a period function to each Maass cusp form. We follow the approach in [7] for weight 0, and adapt it to arbitrary real weights.

4.1. **Poisson kernel.** The function $R(t;z)^s$ in [7, §2.2] can be generalized as the scalar-valued function on $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$

$$R_{s,k}(t,z) \;=\; y^s \left(\frac{t-i}{t-z}\right)^{s-k/2} \left(\frac{t+i}{t-\bar{z}}\right)^{s+k/2}. \tag{4.1}$$

As a function of t it is real-analytic on $\mathbb{P}^1_{\mathbb{R}}$, and as a function of z it is real-analytic on $\mathfrak{H}$. It satisfies

$$\Delta_{-k} R_{s,k}(t,\cdot) \;=\; s(1-s)\, R_{s,k}(t,\cdot)\,. \tag{4.2}$$

For $g \in G$

$$(R_{s,k}|^{\mathrm{ps}}_{s,k} g)|_{-k} g \;=\; (R_{s,k}|_{-k} g)|^{\mathrm{ps}}_{s,k} g = R_{s,k}\,. \tag{4.3}$$

To see this we note that the operator $|^{\mathrm{ps}}_{s,k} g$ acts on the variable t, and the operator $|_{-k} g$ on the variable z. Hence these operators commute. We carry out the computation for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ near the unit element of G. Then the handling of powers of complex quantities is not hard. The relation extends by analyticity. The action of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is multiplication by $e^{-\pi ik}$ for $|^{\mathrm{ps}}_{s,k}$ and multiplication by $e^{\pi ik}$ for $|_{-k}$; so there is no difficulty on the region of discontinuity.

We write $j_k(g,z) = (c_g z + d_g)^{-k}$ and $J^{\mathrm{ps}}_{s,k}(g,t)$ for the factor before $\varphi(gt)$ in (3.1). With (4.3) we see for $\gamma \in \Gamma$:

$$\begin{aligned} v_k(\gamma)^{-1} J^{\mathrm{ps}}_{s,k}(\gamma,t)\, v_{-k}(\gamma)^{-1}\, j_{-k}(\gamma,z)\, R_{s,k}(\gamma t,\gamma z) &= R_{s,k}(t,z)\,,\\ v_k(\gamma)^{-1} J^{\mathrm{ps}}_{s,k}(\gamma,t)\, (R_{s,k}(\gamma t,\cdot)|_{v_{-k},-k}\gamma)(z) &= R_{s,k}(t,z)\,,\\ v_k(\gamma)^{-1} J^{\mathrm{ps}}_{s,k}(\gamma,t)\, R_{s,k}(\gamma t,z) &= R_{s,k}(t,\cdot)|_{v_{-k},-k}\gamma^{-1}(z)\,,\\ R_{s,k}(\cdot,z)|^{\mathrm{ps}}_{v_k,s,k}\gamma\;(t) &= R_{s,k}(t,\cdot)|_{v_{-k},-k}\gamma^{-1}\,(z)\,. \end{aligned} \tag{4.4}$$

4.2. **Green's form.** The generalization of the differential form $[u,v]$ in [7, (1.9)] is

$$\begin{aligned} [u_1,u_2]_k &= \Big(\frac{\partial u_1}{\partial z}\, u_2 + \frac{k}{4iy} u_1 u_2\Big)\, dz + \Big(u_1\, \frac{\partial u_2}{\partial \bar z} - \frac{k}{4iy} u_1 u_2\Big)\, d\bar z\\ &= -2i\Big((X_{+,k}u_1)v_2\, dz + u_1 (X_{-,-k}v_2)\, d\bar z\Big)\,, \end{aligned} \tag{4.5}$$

for $u_1,u_2 \in C^\infty(\mathfrak{H})$ (or for smooth functions on an open subset of $\mathfrak{H}$).

Some properties are

$$d(u_1u_2) = [u_1,u_2]_k + [u_2,u_1]_{-k}\,, \tag{4.6}$$

$$d[u_1,u_2]_k = (u_1\, \Delta_{-k}u_2 - u_2\, \Delta_k u_1)\, \frac{dz\, d\bar z}{-4y^2}\,. \tag{4.7}$$

If u_1 is an eigenvector of Δ_k and u_2 is an eigenvector of Δ_{-k} with the same eigenvalue, then $[u_1,u_2]_k$ is a closed 1-form. For all $g \in G$ we have

$$[u_1|_k g, u_2|_{-k} g]_k = [u_1,u_2]_k \circ g\,, \tag{4.8}$$

where $\circ g$ means the substitution $z \mapsto gz$. We can write it as $[u_1,u_2]_k|_0 g$.

These properties go through if one of u_1 and u_2 is vector-valued. Then the products involved in the formulas make sense, and the relations hold for each component of the vector-valued function.

Let u_1 be vector-valued with values in X_ρ, and let u_2 be scalar-valued. Then we have for $\gamma \in \Gamma$

$$[u_1|_{\rho v_k,k}\gamma, u_2|_{v_{-k},-k}\gamma]_k = \rho(\gamma)^{-1}\, [u_1,u_2]_k \circ \gamma\,. \tag{4.9}$$

Disk model. The upper half-plane is isomorphic as a complex variety with the unit disk by the map $z \mapsto w = \frac{z-i}{z+i}$ with inverse $w \mapsto z = i\frac{1+w}{1-w}$. In the proof of Proposition 6.3 it will be convenient to use the formulation of the Green's form on the unit disk:

$$\begin{aligned} [a,b]_k = &\Big(\frac{\partial a}{\partial w} b + \frac{k(1-\bar w)}{2(1-w)(1-|w|^2)} ab\Big)\, dw\\ &+ \Big(a\frac{\partial b}{\partial \bar w} + \frac{k(1-w)}{2(1-\bar w)(1-|w|^2)} ab\Big)\, d\bar w\,. \end{aligned} \tag{4.10}$$

4.3. **Differential form.** For smooth functions $u\colon \mathfrak{H} \to X_\rho$ we have a differential form of degree 1 with values in the real-analytic functions $\mathbb{P}^1_\mathbb{R} \to X_\rho$:

$$\eta_{s,k}(u) = [u, R_{s,k}]_k . \tag{4.11}$$

This is a differential form on $\mathfrak{H}$ with values in the functions on $\mathbb{P}^1_\mathbb{R}$. If we want to stress the role of the variables, we write $\eta_{s,k}(u; z, t) = [u(z), R_{s,k}(t, z)]_k$. If $\Delta_k u = s(1-s)\, u$, then $\eta_{s,k}(u)$ is a closed form. (Use (4.7) and (4.2).)

For Maass cusp forms $u \in \mathcal{A}^0_k(s, \rho v_k)$, we have for $z_1, z_2 \in \mathfrak{H}$ and for $\gamma \in \Gamma$

$$\int_{\gamma^{-1}z_1}^{\gamma^{-1}z_2} \eta_{s,k}(u) = \int_{z_1}^{z_2} \eta_{s,k}(u)|^{\mathrm{ps}}_{\rho v_k, s,k}\gamma \tag{4.12}$$

as can be checked as follows:

$$\begin{aligned}
\int_{z_1}^{z_2} \eta_{s,k}(u)|^{\mathrm{ps}}_{\rho v_k,s,k}\gamma &= \rho(\gamma)^{-1}\, v_k(\gamma)^{-1} \int_{z_1}^{z_2} \Big[u, R_{s,k}\, \Big|^{\mathrm{ps}}_{s,k}\, \gamma\Big]_k && \text{by (3.2)}\\
&= \rho(\gamma)^{-1}\, v_k(\gamma)^{-1} \int_{z_1}^{z_2} \Big[u, R_{s,k}\, \Big|_{-k}\, \gamma^{-1}\Big]_k && \text{by (4.4)}\\
&= \rho(\gamma)^{-1}\, v_k(\gamma)^{-1} \int_{z_1}^{z_2} \Big[u|_{\rho v_k,k}\gamma^{-1}, R_{s,k}\, \Big|_{-k}\, \gamma^{-1}\Big]_k && \text{since } u \in \mathcal{A}^0_k(s, \rho v_k)\\
&= \int_{z_1}^{z_2} \Big[u\, \Big|_k\, \gamma^{-1}, R_{s,k}\, \Big|_{-k}\, \gamma^{-1}\Big]_k && \text{by (2.7) and}\\
& && (\rho v_k)(\gamma^{-1})^{-1} = (\rho v_k)(\gamma)\\
&= \int_{z_1}^{z_2} [u, R_{s,k}]_k \circ \gamma^{-1} && \text{by (4.9)}\\
&= \int_{\gamma^{-1}z_1}^{\gamma^{-1}z_2} \eta_{s,k}(u)\,.
\end{aligned}$$

4.4. **Cocycles attached to Maass cusp forms.** For $u \in \mathcal{A}^0_k(s, \rho v_k)$ we put

$$c^u(z_1, z_2) = \int_{z_1}^{z_2} \eta_{s,k}(u) \qquad \text{for } z_1, z_2 \in \mathfrak{H}\,. \tag{4.13}$$

This function on $\mathfrak{H} \times \mathfrak{H}$ has values in the Γ-module $\mathcal{V}^\omega_{\rho v_k,s,k}(\mathbb{P}^1_\mathbb{R})$, and it does not depend on the choice of the path from z_1 to z_2. It satisfies the homogeneous cocycle relations

$$\begin{aligned}
c^u(z_1, z_2) + c^u(z_2, z_3) &= c^u(z_1, z_3) && \text{for } z_1, z_2, z_3 \in \mathfrak{H}\,,\\
c^u(\gamma^{-1}z_1, \gamma^{-1}z_2) &= c^u(z_1, z_2)|^{\mathrm{ps}}_{\rho v_k,s,k}\gamma && \text{for } z_1, z_2 \in \mathfrak{H},\ \gamma \in \Gamma\,.
\end{aligned} \tag{4.14}$$

So c^u is a cocycle in $Z^1(\Gamma; \mathcal{V}^\omega_{\rho v_k,s,k}(\mathbb{P}^1_\mathbb{R}))$. (See the discussion in [7, §6.1].) This definition does not need a growth condition, and it works for more automorphic forms than cusp forms.

Parabolic cocycles. If u is a cusp form, then it has exponential decay as $y \uparrow \infty$, and the same holds for its derivatives. This implies that $\int_{z_1}^{\infty} \eta_{s,k}(u)$ converges absolutely and does not depend on the path from $z_1 \in \mathfrak{H}$ to ∞.

The cusps of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ form the set $\mathbb{P}^1_{\mathbb{Q}} \subset \mathbb{P}^1_{\mathbb{R}}$. Each cusp ξ is of the form $\xi = \gamma\infty$ for some (non-unique) $\gamma \in \Gamma$. The invariance of u under $|_{v_k,k}\Gamma$ implies that $\eta_{s,k}(u)$ has fast decay when approaching any cusp of Γ. So we can form integrals $\int_{z_1}^{\xi} \eta_{s,k}(u)$ for any cusp ξ, and also integrals between two cusps. In this way we get

$$c^u_{\mathrm{par}}(\xi_1,\xi_2) = \int_{\xi_1}^{\xi_2} \eta_{s,k}(u) \qquad \text{for } \xi_1,\xi_2 \in \mathbb{P}^1_{\mathbb{Q}}\,. \tag{4.15}$$

The function c^u_{par} on $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ has properties analogous to (4.14). It requires some work to determine its regularity properties. Working out $[u, R_{s,k}(t,\cdot)]_k$ we see that $c^u_{\mathrm{par}}(\xi_1,\xi_2)$ is real-analytic at all points of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1,\xi_2\}$.

The behavior at $t = \xi_1$ and $t = \xi_2$ has to be considered. By the transformation behavior under Γ we can reduce the consideration to integrals $\int_z^\infty \eta_{s,k}(u)$. Proceeding in the same way as in [7, Proposition 9.7] we can show that it is C^∞ in a neighborhood of ∞ in $\mathbb{P}^1_{\mathbb{R}}$. Here we are content to have continuity.

In a notation analogous to the notations in [7], we define the Γ-module $\mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ as the space of functions in the space $\mathcal{V}^{0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ of continuous functions that restrict to an element of $\mathcal{V}^{\omega}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}} \smallsetminus E)$ for a finite set $E \subset \mathbb{P}^1_{\mathbb{Q}}$. For $c^u_{\mathrm{par}}(\xi_1,\xi_2)$ the set E can be taken as $\{\xi_1,\xi_2\}$.

The index *par* in

$$c^u_{\mathrm{par}} \in Z^1_{\mathrm{par}}(\Gamma; \mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})) \tag{4.16}$$

indicates *parabolic*. For each $\xi \in \mathbb{P}^1_{\mathbb{Q}}$ there is an infinite subgroup of Γ fixing ξ. For $\xi = \infty$ this is the subgroup generated by T and $-I_2$. This has the consequence that cocycles on $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ do not compute the usual cohomology groups $H^1(\Gamma;\cdot)$, but the *parabolic cohomology groups* $H^1_{\mathrm{par}}(\Gamma;\cdot)$.

4.5. **The period function of a Maass cusp form.**

Proposition 4.1. *Let $u \in \mathcal{A}^0_k(s,\rho v_k)$. We put $P(u) = c^u_{\mathrm{par}}(0,\infty)$.*

(a) *$P(u)$ has values in the 1-eigenspace of $|^{\mathrm{ps}}_{\rho v_k,s,k}(-I_2)$.*
(b) *$P(u) = -P(u)|^{\mathrm{ps}}_{\rho v_k,s,k}S = P(u)|^{\mathrm{ps}}_{\rho v_k,s,k}T + P(u)|^{\mathrm{ps}}_{\rho v_k,s,k}T'$.*
(c) *$P(u)$ is real-analytic on $(\infty,0)_c \cup (0,\infty)$, with a continuous extension across 0 and ∞.*
(d) *$P(u)$ has a holomorphic extension to $\mathbb{C} \smallsetminus i\mathbb{R}$.*
(e) *If $\xi_1,\xi_2 \in \mathbb{P}^1_{\mathbb{Q}}$, then there is a finite number of elements $\gamma_j \in \Gamma$ such that*

$$c^u_{\mathrm{par}}(\xi_1,\xi_2) = \sum_j P(u)|^{\mathrm{ps}}_{\rho v_k,s,k}\gamma_j\,. \tag{4.17}$$

Proof. Statements (a) and (c) are specializations of properties already observed for integrals $c^u_{\mathrm{par}}(\xi_1,\xi_2)$ with general ξ_1 and ξ_1 in $\mathbb{P}^1_{\mathbb{Q}}$.

We can take the path of integration from 0 to ∞ for $P(u)$ as the positive imaginary axis. Then we obtain part (d).

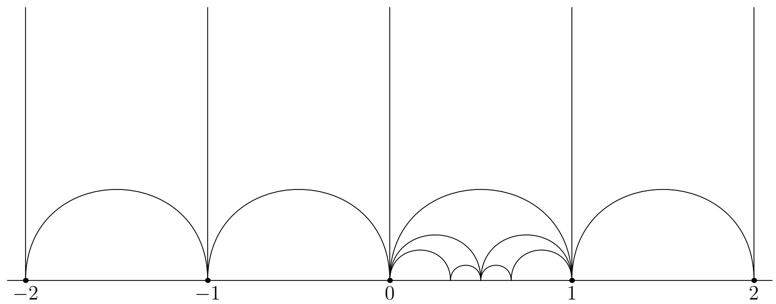


FIGURE 1. The Farey tesselation.
This tesselation of $\mathfrak{H}$ consists of all Γ translates of the hyperbolic triangle with corner points 0, ∞, and 1.

By (4.12)

$$\int_0^\infty \eta_{s,k}(u)|^{\mathrm{ps}}_{\rho v_k,s,k} S = \int_\infty^0 \eta_{s,k}(u) = -\int_0^\infty \eta_{s,k}(u)\,.$$

This gives the first relation in (b). For the other relation we use

$$\int_0^\infty \eta_{s,k}(u)|^{\mathrm{ps}}_{\rho v_k,s,k}(T+T') = \int_{-1}^\infty \eta_{s,k}(u) + \int_0^{-1} \eta_{s,k}(u) = \int_0^\infty \eta_{s,k}(u)\,.$$

We use the well-known Farey tesselation (sketched in Figure 1). The endpoints of the edges run through $\mathbb{P}^1_\mathbb{Q}$. Each edge is the translate $\gamma e_{0,\infty}$ for some $\gamma \in \Gamma$, where $e_{0,\infty}$ denotes the path from 0 to ∞. We note that $e_{\infty,0} = S^{-1} e_{0,\infty}$. Each vertex is connected to ∞ by a path along finitely many edges of the tesselation. (Use a Farey sequence with bounded denominators to see this.) In this way we get for each pair $(\xi_1, \xi_2) \in \mathbb{P}^1_\mathbb{Q}$ a finite path $\sum_j \gamma_j\, e_{0,\infty}$ from ξ_1 to ξ_2. We use this to obtain part (e). □

Proposition 4.2 (Period function of Maass cusp form)**.**

(i) *For $P(u)$ as in Proposition 4.1, its restriction to $(0,\infty)$ defines $\mathrm{pf}(u)$ by*

$$\mathrm{pf}(u) := P(u)|_{(0,\infty)} \tag{4.18}$$

This determines a linear map $\mathrm{pf}\colon \mathcal{A}^0_k(s, \rho v_k) \to \mathsf{FE}^\omega_{\rho v_k,s,k}$. We call $\mathrm{pf}(u)$ the period function *associated to u.*

(ii) *For each $f \in \mathsf{FE}^\omega_{\rho v_k,s,k}$ there is a unique element $p \in \mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_\mathbb{R})$ for which properties (a), (b) and (c) in Proposition 4.1 hold, with restriction $p|_{(0,\infty)}$ equal to f.*

Proof. Definition 3.1 of the space $\mathsf{FE}^\omega_{\rho v_k,s,k}$ of period functions has been arranged in such a way that the restriction of $P(u)$ to $(0,\infty)$ is a period function.

Conversely, any period function f has a unique extension $p \in \mathcal{V}^\omega_{s,k}(\mathbb{R} \smallsetminus \{0\})$ satisfying $p|_{\rho v_k,s,k} S = -p$. The limits in condition (c) in Definition 3.1 imply that p extends as a continuous function on $\mathbb{P}^1_\mathbb{R}$, hence $p \in \mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_\mathbb{R})$. Separate computations on $(\infty,-1)_c$ and $(-1,0)$ show that p satisfies the three term relation. The extension to $\mathbb{C}'$ given in Proposition 3.2 shows that f extends holomorphically to the right half-plane. Then the action of S gives the holomorphy on the left half-plane. □

The cocycle c^u_{par} on $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ has the property that $c^u_{\mathrm{par}}(0,\infty)$ cannot be changed by adding a coboundary db where $b\colon \mathbb{P}^1_{\mathbb{Q}} \to \mathcal{V}^{\omega^0,0}_{\rho\upsilon_k,s,k}$ is equivariant under Γ, as the following lemma indicates:

Lemma 4.3. *Let $s \neq 0$. Each $f \in \mathcal{V}^{\omega^0,\infty}_{\rho\upsilon_k,s,k}$ is analytic on (α,∞) for some $\alpha \geq 0$. If $f|^{\mathrm{ps}}_{\rho\upsilon_k,s,k}T = f$ on (α,∞), then $f = 0$.*

Proof. The first statement follows from the fact that f is real-analytic on $\mathbb{P}^1_{\mathbb{R}}$ except at finitely many cusps of Γ.

Let f_l be a component function of f in the decomposition in (2.11). For all $t > \alpha$

$$\begin{aligned} f_l(t) &= e^{2\pi i m\kappa_l}\Big(\frac{t-i}{t-i+m}\Big)^{s-k/2}\Big(\frac{t+i}{t+i+m}\Big)^{s+k/2} f_l(t+m) && \text{for } m \in \mathbb{Z}_{\geq 0} \\ &\sim e^{2\pi i m\kappa_l}\, m^{-2s}\,(t^2+1)^{2s}\, f_l(\infty) && \text{as } m \to \infty\,. \end{aligned} \tag{4.19}$$

If $s \neq 0$, then this implies that f is the zero function. □

Role of k. At least for $s \not\equiv \frac{k}{2} \bmod 1$ we know that all spaces $\mathcal{A}^0_{k'}(s,\rho\upsilon_k)$ with $k' \equiv k \bmod 2$ are related by the weight shifting operators; see (2.36). We do not know the effect of the weight shifting operators on the associated period functions.

5. Transfer operators

Proposition 4.2 shows we can associate period function to Maass cusp forms. In the introduction we indicated that discretization of the geodesic flow on the sphere bundle of $\Gamma\backslash\mathfrak{H}$ leads to transfer operators. Here we discuss such transfer operators, and show that their eigenfunctions with eigenvalue 1 lead to period functions.

5.1. **Slow transfer operator.** We denote by $C^\omega_\rho(I)$ the space of real-analytic functions on the interval $I \subset \mathbb{P}^1_{\mathbb{R}}$ with values in the 1-eigenspace of $|^{\mathrm{ps}}_{\rho\upsilon_k,s,k}(-I_2)$.

The transfer operators that we will discuss act on functions in $C^\infty_\rho(0,\infty)$. Let Γ' be the semigroup in Γ generated by T and T'. Since $\delta^{-1}(0,\infty) \supset (0,\infty)$ for each $\delta \in \Gamma'$, the operator $|^{\mathrm{ps}}_{\rho\upsilon_k,s,k}\delta$ followed by restriction to $(0,\infty)$ is well defined on $C^\omega_\rho(0,\infty)$; it is not a bijection. The restriction is understood in the formulas.

Definition 5.1. The *slow transfer operator* is

$$\mathcal{L}_{\rho\upsilon_k,s,k}\colon C^\omega_\rho(0,\infty) \to C^\omega_\rho(0,\infty)\,, \quad f \mapsto f|^{\mathrm{ps}}_{\rho\upsilon_k,s,k}(T+T')\,.$$

The period functions in Definition 3.1 are 1-eigenfunctions of the slow transfer operator. Since in the definition of $C^\omega_\rho(I)$ there are no conditions on the behavior near the boundary points, there may be many more 1-eigenfunctions of the slow transfer operator than period functions.

5.2. **One-sided averages.** We need results concerning the *Lerch transcendent* in (5.1) below. Proposition 5.2 below was shown in [5], starting from results of Kanemitsu, Katsurada and Oshimoto in [15] and Katsurada [16]. See also [12, Proposition A.1]. Lagarias and Li [17] give further going information on the Lerch transcendent.

Proposition 5.2. *The Lerch transcendent*

$$H(s,\zeta,z) \;=\; \sum_{n\geq 0} \zeta^n \;(z+n)^{-s} \tag{5.1}$$

converges absolutely for $\alpha > 0$, $\operatorname{Re} s > 1$, $|\zeta| \leq 1$.

(i) Meromorphic extension in (s,z)

  (a) *If* $\zeta = 1$, *then* $(s,z) \mapsto H(s,1,z)$ *has a first order singularity along* $s = 1$.

  (b) *If* $|\zeta| = 1$, $\zeta \neq 1$, *then* $(s,z) \mapsto H(s,1,z)$ *is holomorphic on* $\mathbb{C} \times (\mathbb{C} \smallsetminus (-\infty, 0])$.

(ii) Asymptotic behavior. *For* $|\zeta| = 1$ *and* $s \in \mathbb{C}$ *and* $N \in \mathbb{Z}_{\geq 0}$ *there is an expansion*

$$H(s,\zeta,z+1/2) \;=\; \sum_{n=-1}^{N-1} C_n(\zeta,s)\, z^{-n-s} + \mathrm{O}(|z|^{-N-\operatorname{Re} s}) \tag{5.2}$$

on any region $\delta - \pi \leq \arg(z) \leq \pi - \delta$, $0 < \delta \leq \pi$.

Furthermore, if $\zeta \neq 1$, *then*

$$C_{-1}(\zeta,s) \;=\; 0\,. \tag{5.3}$$

The following lemma defines so-called one-sided averages, which we will use to define the fast transfer operator.

Lemma 5.3. *Let* $\alpha, \beta \in \mathbb{R}$ *satisfy* $\beta < \alpha$.

(a) *For all* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > \frac{1}{2}$ *the* one-sided averages

$$\begin{aligned} (f|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{+})(t) \;&=\; \sum_{m=0}^{\infty} (f|^{\mathrm{ps}}_{\rho v_k,s,k} T^m)(t)\,,\\ (f|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{-})(t) \;&=\; -\sum_{m\leq -1} (f|^{\mathrm{ps}}_{\rho v_k,s,k} T^m)(t) \end{aligned} \tag{5.4}$$

converge absolutely for all $f \in C^{\omega}_{\rho}((\alpha,\beta)_c)$ *and define*

$$f|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{+} \in C^{\omega}_{\rho}(\alpha,\infty) \quad \textit{and} \quad f|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{-} \in C^{\omega}_{\rho}((\infty,\beta+1)_c)\,.$$

(b) *The operators* $|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{+}$ *and* $|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{-}$ *commute with* $|^{\mathrm{ps}}_{\rho v_k,s,k}T$.

(c) *For* $f \in C^{\omega}_{\rho}((\alpha,\beta)_c)$ *the function* $f|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{+}|^{\mathrm{ps}}_{\rho v_k,s,k}(1-T)$ *is equal to the restriction of* f *to* (α,∞), *and the function* $f|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{-}|^{\mathrm{ps}}_{\rho v_k,s,k}(1-T)$ *is equal to the restriction of* f *to* $(\infty,\beta)_c$.

(d) *For each* $f \in C^{\omega}((\alpha,\beta)_c)$ *the functions* $(s,t) \mapsto (f|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^{\pm})(t)$ *are real-analytic in* (s,t) *and holomorphic in* s *on the region of* (s,t) *with* $\operatorname{Re} s > \frac{1}{2}$, *and* $t \in (\alpha,\infty)$, *respectively* $t \in (\infty,\beta+1)_c$.

Proof. Similar results are proved earlier, in slightly differing contexts; see e.g., [5, §4] and [9, §7.6].

We note that $\infty \in (\alpha,\beta)_c$. The function f is bounded on a neighborhood of ∞ in $\mathbb{P}^1_{\mathbb{R}}$. Absolute convergence follows directly from

$$(5.5)\quad (f|^{\mathrm{ps}}_{\rho v_k,s,k}T^m)(t) = \Big(\frac{t-i}{t-i+m}\Big)^{s-k/2}\Big(\frac{t+i}{t+i+m}\Big)^{s+k/2} e^{-\pi i m k/6}\,\rho(T^{-m})f(t+m)$$

and the fact that the eigenvalues of the unitary operator $\rho(T)$ in X_ρ have absolute value 1. The other statements follow by rearranging the order of the infinite sums, and the observation that the absolute convergence is uniform for (s,t) in compact sets. □

Proposition 5.4. *Let $f \in C^\omega((0,-1)_c)$.*

(i) *The functions $(s,t) \mapsto (f|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^\pm)(t)$ extend as real-analytic functions on $\{(s,t)\in\mathbb{C}\times(0,\infty)\}$, respectively $\{(s,t)\in\mathbb{C}\times(\infty,0)_c\}$ that are meromorphic in s with at most first order singularities in $s=\frac{n}{2}$ with $n\in\mathbb{Z}_{\le 1}$.*

A singularity at $s=\frac12$ occurs if and only if there exists an eigenvector e_l of $\rho(T)$ with $\kappa_l=0$ and $\big(f(\infty),e_l\big)_\rho \neq 0$.

(ii) *The assertions in (b), (c) and (d) in Lemma 5.3 stay valid for the extensions.*

(iii) *We apply to f the decomposition (2.11). There are asymptotic expansions of the form*

$$(5.6)\quad \begin{aligned}(f_l|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^+)(t) &\sim \sum_{n=-1}^{N-1} C_{n,l}(s)t^{-n} + \mathrm{O}(t^{-N}) &&\text{as } t\uparrow\infty\,,\\ (f_l|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^-)(t) &\sim \sum_{n=-1}^{N-1} C_{n,l}(s)t^{-n} + \mathrm{O}(t^{-N}) &&\text{as } t\downarrow-\infty\,,\end{aligned}$$

for each $N\in\mathbb{Z}_{\ge 0}$.

Proof. The general approach in [5, 7, 9] goes through with some adaptations.

We have to work with the components f_l in the decomposition (2.11), and if $\kappa_l \neq 0$, the factor $e^{-2\pi i m\kappa_l}$ lead to the Lerch transcendent instead of the Hurwitz zeta function. The factors $\big(\frac{t-i}{t-i+m}\big)^{s-k/2}$ and $\big(\frac{t+i}{t+i+m}\big)^{s+k/2}$ have a more complicated expansion in terms of powers of $t+m$. Taking this into account, we can follow the approach in [5, §4.2] to prove the theorem. □

5.3. **Fast transfer operator.** Let $f\in C^\omega_\rho(0,\infty)$. Then the function $f|^{\mathrm{ps}}_{\rho v_k,s,k}T'$ is a function in $C^\omega((0,-1)_c)$. So this function satisfies the condition in Lemma 5.3, and we can form the *fast transfer operator*

$$(5.7)\quad \mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k}f = (f|^{\mathrm{ps}}_{\rho v_k,s,k}T')|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^+ = \sum_{n\ge 0} f|^{\mathrm{ps}}_{\rho v_k,s,k}T'T^n\,.$$

Proposition 5.5. *The series in the definition of $\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k}f$ in (5.7) converges absolutely for* $\mathrm{Re}\, s>\frac12$.

(i) *The family* $s \mapsto f|^{\mathrm{ps}}_{\rho v_k,s,k}$ *extends meromorphically to* $s \in \mathbb{C}$ *with at most first order singularities at points of* $\frac{1}{2}\mathbb{Z}_{\leq 1}$.

A first order singularity occurs at $s = \frac{1}{2}$ *if and only if there is an eigenvector* e_l *of* $v_k(T)\,\rho(T)$ *with* $\kappa_l = 0$ *for which*

$$\Big(\rho(T')^{-1}\, v_k(T')^{-1} f(1), e_l\Big)_\rho \neq 0\,. \tag{5.8}$$

(ii) *For all values of* s *for which* $\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k} f$ *is holomorphic it defines a function in* $C^\omega(0,\infty)$ *with an asymptotic behavior as indicated in* (5.6).

(iii) *For all* $f \in C^\omega_\rho$

$$(\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k} f)|^{\mathrm{ps}}_{\rho v_k,s,k}(1-T) \;=\; f|^{\mathrm{ps}}_{\rho v_k,s,k} T' \;=\; (\mathcal{L}_{\rho v_k,s,k} f)|^{\mathrm{ps}}_{\rho v_k,s,k}(1-T)\,. \tag{5.9}$$

(iv) *If* f *is a* 1*-eigenfunction of* $\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k}$ *then* f *is a* 1*-eigenfunction of* $\mathcal{L}_{\rho v_k,s,k}$.

Proof. Most of these assertions follow directly from Proposition 5.4. The relations in (5.9) and part (iv) follow from Definition 5.1 and assertion (c) in Lemma 5.3. □

Proposition 5.6. *Let* $s \in \mathbb{C} \smallsetminus \frac{1}{2}\mathbb{Z}_{\leq 0}$. *If* $f \in \mathsf{FE}^\omega_{\rho v_k,s,k}$, *then* f *is a* 1*-eigenfunction of the fast transfer operator* $\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k}$.

Proof. Let $f \in \mathsf{FE}^\omega_{\rho v_k,s,k}$. Lemma 3.3 and part (i) of Proposition 5.5 show that the fast transfer operator $\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k} f$ is holomorphic at $s = \frac{1}{2}$.

By part (b) in Definition 3.1 and (5.9) we have

$$\Big(\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k} f - f\Big)|^{\mathrm{ps}}_{\rho v_k,s,k}(1-T) \;=\; 0\,.$$

So the difference $p = \mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k} f - f$ is invariant under $|^{\mathrm{ps}}_{\rho v_k,s,k} T$. Like in the proof of Lemma 4.3 we have, now for $t \in (0,\infty)$,

$$p(t) \;=\; v_k(T)^q\, \rho(T)^q \Big(\frac{t-i}{t-i+q}\Big)^{s-k/2} \Big(\frac{t+i}{t+i+q}\Big)^{s+k/2} p(t+q)\,, \tag{5.10}$$

for all $q \in \mathbb{Z}$. The limit as $t \uparrow \infty$ of f exists by condition (c) in Definition 3.1, and the expansion of $\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k} f$ can have a term with t^1, see (5.6). Thus, if $p \neq 0$ then it satisfies $p(t) \sim A_m t^m$ as $t \uparrow \infty$ for some $m \in \mathbb{Z}_{\leq 1}$ and some $A_m \neq 0$. We go over to the eigendecomposition (2.11). Taking $t_0 \in (0,\infty)$ such that $p_l(t_0) \neq 0$ we have

$$\begin{aligned} p_l(t_0) \;&=\; e^{2\pi i q \kappa_l} \Big(\frac{t_0-i}{t_0-i+q}\Big)^{s-k/2} \Big(\frac{t_0+i}{t_0+i+q}\Big)^{s+k/2} p_l(t_0+m) \\ &\sim\; (t_0-i)^{s-k/2}(t_0+i)^{s+k/2} e^{2\pi i q\kappa_l}\, q^{-2s}\, A_m\,(t_0+q)^m \\ &\sim\; (t_0-i)^{s-k/2}(t_0+i)^{s+k/2}\, A_m\, e^{2\pi i q \kappa_l}\, q^{m-2s} \qquad \text{as } q \uparrow \infty\,. \end{aligned} \tag{5.11}$$

This is possible only if $s = \frac{m}{2} \in \frac{1}{2}\mathbb{Z}_{\leq 1}$. In the statement of the proposition these values of s are excluded. Hence $p = 0$ and $\mathcal{L}^{\mathrm{fast}}_{\rho v_k,s,k} f = f$. □

6. Analytic boundary germs

The step from parabolic cohomology to Maass forms in [7, §12] is carried out by going over from principal series modules of analytic functions of $\mathbb{P}^1_{\mathbb{R}}$ to isomorphic modules of analytic boundary germs. The latter modules allow us to use the geometry of the upper half-plane to construct Maass forms from cocycles.

6.1. **Kernel function.** In the construction of period functions associated to Maass cusp forms we used the Poisson kernel $R_{s,k}$ defined on $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$. We need to replace it by a kernel function on $\mathfrak{H} \times \mathfrak{H}$ with similar properties.

Here it is useful to work on the universal covering group, discussed in §2.3. We first describe a function $Q_{s,k}$ on $\tilde{G} \smallsetminus \tilde{K}$, where $\tilde{K} = \{\tilde{\mathrm{k}}(\vartheta) \; : \; \vartheta \in \mathbb{R}\}$. We have the *polar decomposition* $\tilde{G} \smallsetminus \tilde{K} = \tilde{K}\tilde{A}_+\tilde{K}$, with $\tilde{A} = \{\tilde{\mathrm{a}}(y) \; : \; y > 0\}$ and $\tilde{A}_+ = \{\tilde{\mathrm{a}}(y) \; : \; y > 1\}$, $\tilde{\mathrm{a}}(y) = \tilde{\mathrm{p}}(iy)$.

Lemma 6.1. *Let $s \in \mathbb{C}$ and $k \in \mathbb{R}$ satisfy $s \notin \frac{1}{2}\mathbb{Z}_{\le 1} \cup \left(-\frac{k}{2} + \mathbb{Z}\right) \cup \left(\frac{k}{2} + \mathbb{Z}\right)$. There is a function $Q_{s,k} \in C^\infty(\tilde{G} \smallsetminus \tilde{K})$ satisfying*

(a) $Q_{s,k}(\tilde{\mathrm{k}}(\vartheta_1)\tilde{\mathrm{a}}(y)\tilde{\mathrm{k}}(\vartheta_2)) = e^{ik(\vartheta_1+\vartheta_2)} Q_{s,k}(\tilde{\mathrm{a}}(y))$.
(b) $\Delta Q_{s,k} = s(1-s)\, Q_{s,k}$.
(c) $Q_{s,k}(\mathrm{a}(y)) = \mathrm{O}(y^{-s})$ *as* $y \to \infty$.
(d) $Q_{s,k}(\tilde{\mathrm{a}}(y)) = -\log(1-v)\, h_1(v) + h_2(v)$ *for* $v = \frac{4y}{(y+1)^2}$, *with C^∞-functions h_1 and h_2 on a neighborhood of* 1 *in* $\mathbb{R}$, *and* $h_1(1) = 1$.
(e) $Q_{s,k}(g^{-1}) = Q_{s,-k}(g)$ *for* $g \in \tilde{G} \smallsetminus \tilde{K}$.

Proof. Functions on $\tilde{G}$ satisfying a generalization of condition (a) are needed to describe the polar expansion of scalar-valued Maass forms at the point $i \in \mathfrak{H}$. They are given in [3, 4.2.6 and 4.2.9] in terms of $u = \frac{(y-1)^2}{4y}$. Condition (b) imposes a hypergeometric differential equation, with a two-dimensional solutions space. For the expansion of Maass forms we need a solution that is C^∞ at $y = 1$. Here we need a solution with a singularity at $y = 1$ that is small for $\mathrm{Re}\, s \ge \frac{1}{2}$ as $y \uparrow \infty$ and $y \downarrow 0$. A multiple of the solution $\mu_k(i, ks + 1/2)$ in [3, p. 4.2.6] is the one that we need here.

Using (a) and (b) we obtain

$$
\begin{aligned}
Q_{s,k}(\tilde{\mathrm{k}}(\vartheta_1)\tilde{\mathrm{a}}(y)\tilde{\mathrm{k}}(\vartheta_2)) &= \frac{\Gamma(s-k/2)\,\Gamma(s+k/2)}{\Gamma(2s)}\, e^{ik(\vartheta_1+\vartheta_2)} \\
&\quad \cdot v^s\, {}_2F_1\left[\begin{matrix} s-k/2,\, s+k/2 \\ 2s \end{matrix} \,\Big|\, v\right], \qquad v = \frac{4y}{(y+1)^2}\,.
\end{aligned} \tag{6.1}
$$

The singularities of the solution are avoided by the condition on s and k in the lemma. Since the hypergeometric function is holomorphic at $v = 0$ with value 1, we get property (c). There is a logarithmic singularity at $v = 1$. The gamma factors have been chosen such that the hypergeometric function is $-\log(1-v)$ as $v \uparrow 1$. This leads to assertion (d).

The function $Q_{s,k}(\tilde{\mathrm{a}}(y))$ is invariant under $y \mapsto 1/y$. With

$$
\left(\tilde{\mathrm{k}}(\vartheta_1)\tilde{\mathrm{a}}(y)\tilde{\mathrm{k}}(\vartheta_2)\right)^{-1} = \tilde{\mathrm{k}}(-\vartheta_2)\tilde{\mathrm{a}}(1/y)\tilde{\mathrm{k}}(-\vartheta_1)
$$

this implies assertion (e). □

Proposition 6.2. *Let $s \in \mathbb{C}$, $k \in \mathbb{R}$, and $s \notin \frac{1}{2}\mathbb{Z}_{\leq 1} \cup \left(-\frac{k}{2}+\mathbb{Z}\right) \cup \left(\frac{k}{2}+\mathbb{Z}\right)$. There is a kernel function $q_{s,k}$ with the following properties:*

(i) $q_{s,k} \in C^\infty\Bigl(\bigl\{(z_1, z_2 \in \mathfrak{H}^2\ ;\ z_1 \neq z_2\bigr\}\Bigr)$,
(ii) $\Delta_k q_{s,k}(z_1,\cdot) = s(1-s)\, q_{s,k}(z_1,\cdot)$, *and* $\Delta_{-k} q_{s,k}(\cdot, z_2) = s(1-s)\, q_{s,k}(\cdot, z_2)$,
(iii) $q_{s,k}(z_2, z_1) = q_{s,-k}(z_1, z_2)$,
(iv) $q_{s,k}\bigl(|_{-k} \times |_k\bigr) g = q_{s,k}$ *for all $g \in G$. Here $|_{-k} g$ acts on the first variable and $|_k g$ on the second variable.*

Proof. We take for $z_1 \neq z_2 \in \mathfrak{H}$

$$q_{s,k}(z_1, z_2) = Q_{s,k}(\tilde{\mathrm{p}}(z_1)^{-1}\tilde{\mathrm{p}}(z_2)), \tag{6.2}$$

with $\tilde{\mathrm{p}}(z) \in \tilde{G}$ as discussed in §2.3. This satisfies assertion (i). Relation (iii) follows from Lemma 6.1(e).

The differential operator Δ in (2.32) commutes with left translation, and corresponds to Δ_k in (2.1) on functions in weight k. This implies that $\Delta_k q_{s,k}(z_1,\cdot) = s(1-s) q_{s,k}(z_1,\cdot)$. With (iii) this implies $\Delta_{-k} q_{s,k}(\cdot, z_2) = s(1-s) q_{s,k}(\cdot, z_2)$ as well.

We have for all $\tilde{g} \in \tilde{G}$

$$Q_{s,k}(\tilde{\mathrm{p}}(z_1)^{-1}\tilde{\mathrm{p}}(z_2)) = Q_{s,k}\Bigl((\tilde{g}\tilde{\mathrm{p}}(z_1))^{-1}\tilde{g}\tilde{\mathrm{p}}(z_2)\Bigr).$$

Hence assertion (iv) follows from (2.29). □

Use of the disk model. Let $z_1 \in \mathfrak{H}$ be near to i and $z_2 = i$. Then $w_1 = \frac{z_1 - i}{z_2 + i}$ is near to 0. Taking $\vartheta_1 = \frac{1}{2}\arg(w_1)$ and $\vartheta_2 = \vartheta_1 - \frac{\pi}{2} + \arg(z+i)$ we can check that

$$\tilde{\mathrm{p}}(z_1) = \tilde{\mathrm{k}}(\vartheta_1)\tilde{\mathrm{a}}(t)\tilde{\mathrm{k}}(\vartheta_2), \tag{6.3}$$

with $t = \frac{1+|w_1|}{1-|w_1|}$. Then $\tilde{\mathrm{p}}(z_1) = \tilde{\mathrm{k}}(\vartheta_1)\tilde{\mathrm{a}}(t)\tilde{\mathrm{k}}(\vartheta_2)$ with $\vartheta_1 + \vartheta_2 = \frac{\pi}{2} - \arg(z+i)$.

Hence

$$q_{s,k}(z_1, i) = Q_{s,-k}(\tilde{\mathrm{p}}(z)1)) = e^{-ik(\vartheta_1+\vartheta_2)}\, Q_{s,-k}\Bigl(\frac{1+|w_1|}{1-|w_1|}\Bigr). \tag{6.4}$$

6.2. **Integration with the kernel function.** We generalize the integral formula in [7, Theorem 1.1], proved in [6, Theorem 3.1] (quoted in [7] as Theorem 2.1).

Proposition 6.3. *Let C be a piecewise smooth positively oriented simple closed curve in $\mathfrak{H}$ and let U be an open region in $\mathfrak{H}$ containing the curve C and its interior. If $u \in C^\infty(U)$ satisfies $\Delta_k u = s(1-s)u$, then for $z_2 \in \mathfrak{H} \setminus C$*

$$\int_C [u, q_{s,k}(\cdot, z_2)]_k = \begin{cases} 2\pi i\, u(z_2) & \text{if } z_2 \text{ is inside } C, \\ 0 & \text{if } z_2 \text{ is outside } C. \end{cases} \tag{6.5}$$

Proof. With (4.5) we have a differential form $[u, q_{s,k}(\cdot, z_2)]_k$ on $U \setminus \{z_2\}$, which is closed by (4.7) and Proposition 6.2(ii). The integral

$$\int_C [u, q_{s,k}(\cdot, z_2)]_k$$

does not change if we deform the path C continuously in $U \smallsetminus \{z_2\}$. In particular, the integral is zero if z_2 is outside C. We proceed under the assumption that z_2 is inside C.

Let us take $g = \mathrm{p}(z_2) \in G$, with $\mathrm{p}(x+iy) = \mathrm{pr}\,\tilde{\mathrm{p}}(x+iy) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{smallmatrix}\right)$. Then $z_2 = gi$, and $C_1 = g^{-1}C$ encircles i once, contained in the open set $g^{-1}U$ containing i. With (4.8) we obtain

$$\int_C [u, q_{s,k}(\cdot, z_2)]_k \;=\; \int_{C_1} [u|_k g, q_{s,k}(\cdot, i)]_k \,. \tag{6.6}$$

We shrink the curve C_1 to a small hyperbolic circle around i.

We use the disk model, with coordinate $w = \frac{z-i}{z+i}$. Then we can take C_1 as a circle around $w = 0$ with radius $\mathrm{O}(\varepsilon)$ and let $\varepsilon \downarrow 0$. We use the description (4.10) for the Green's form. The function a corresponds to u, and the function b to $z_1 \mapsto q_{s,k}(z_1, i)$. By Lemma 6.1(d) we obtain

$$\begin{aligned} b &= (1-w)^{k/2}(1-\bar{w})^{-k/2}\Big(-h_1(1-w\bar{w})\log(w\bar{w}) + h_2(1-w\bar{w})\Big) = \mathrm{O}(\log\varepsilon)\,, \\ \partial_{\bar{w}} b &= \frac{-1}{2\bar{w}}(1-w)^{-k/2}(1-\bar{w})^{k/2-1}\Big(k\bar{w}h_2(1-|w|^2) \\ &\quad - h_1(1-|w|^2)(k\bar{w}\log|w|^2 + 2\bar{w} - 2) \\ &\quad - 2|w|^2(1-\bar{w})(h_1'(1-|w|^2)\log|w|^2 - h_2'(1-|w|^2))\Big) \\ &= \frac{-1}{2}\,\bar{w}^{-1}\Big(2 + \mathrm{O}(\varepsilon\log\varepsilon)\Big)\,. \end{aligned}$$

We write $w = \varepsilon e^{i\varphi}$. The first term in (4.10) is

$$\Big((\partial_w a)\,b + \frac{k(1-\bar{w})}{2(1-w)(1-|w|^2)}ab\Big)\,dw \;=\; \mathrm{O}(\log\varepsilon) i\varepsilon\, e^{i\varphi}\,d\varphi = o(1)\,, \tag{6.7}$$

and does not contribute to the integral. The second term is

$$\begin{aligned} &\Big(a\,(\partial_{\bar{w}} b) + \frac{k(1-w)}{2(1-\bar{w})(1-|w|^2}ab\Big)\,d\bar{w} \\ &\qquad = \Big(-a\,\varepsilon^{-1}e^{i\varphi} + a\,\mathrm{O}(\log\varepsilon) + \mathrm{O}(\log\varepsilon)\Big)(-i\varepsilon)e^{-i\varphi}\,d\varphi \end{aligned}$$

This gives in the limit $\varepsilon \downarrow 0$ for the total integral the value

$$2\pi\,a(0) = 2\pi i\,(u|_l\tilde{\mathrm{p}}(z_2))(i) \;=\; 2\pi i\, e^{k\cdot 0}u(z_2)\,.$$ □

6.3. **Boundary germs.** In Proposition 4.2 we associated to a Maass cusp form in $\mathcal{A}_k^0(s, \rho v_k)$ a 1-cocycle c^u_{par} on $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ with values in the module $\mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$, which contains $\mathcal{V}^{\omega}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ and is contained in $\mathcal{V}^{0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$. To go back from period functions to Maass cusp forms we go over to modules of boundary germs that are isomorphic to the principal series modules $\mathcal{V}^{\omega}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$.

Sheaves related to eigenfunctions of Δ_k. For each open set $\Omega \subset \mathfrak{H}$ we put

$$\mathcal{E}_{s,k}(\Omega) \;=\; \{f \in C^\infty(\Omega) \;:\; \Delta_k f = s(1-s)\,f\}\,. \tag{6.8}$$

This defines a sheaf $\mathcal{E}_{s,k}$ on $\mathfrak{H}$.

We turn to subsets $\Omega \subset \mathbb{P}^1_{\mathbb{C}}$ that have a non-empty intersection with $\mathfrak{H}$. We put $\mathcal{B}_{s,k}(\Omega) = \mathcal{E}_{s,k}(\Omega)$ if $\Omega \subset \mathfrak{H}$, and if $\Omega \cap \mathbb{P}^1_{\mathbb{R}} \neq \emptyset$, then we put

$$\begin{aligned} \mathcal{B}_{s,k}(\Omega) &= \Big\{ f \in \mathcal{E}_{s,k}(\Omega \cap \mathfrak{H}) \;:\; \text{the function } F(z) = \Phi_{s,k}(z) f(z) \\ &\qquad \text{on } \Omega \cap \mathfrak{H} \text{ extends to a real-analytic function on } \Omega \Big\}, \\ \Phi_{s,k}(z) &= y^{-s}\,(z+i)^{s+k/2}\,(\bar z - i)^{s-k/2}\,. \end{aligned} \tag{6.9}$$

The argument of $z+i$ is in $[0,\pi]$ for $z \in \mathfrak{H} \cup \mathbb{R}$, and the argument of $\bar z - i$ is in $[-\pi, 0]$. In the coordinate $w = \frac{z-i}{z+i}$

$$\Phi_{s,k}(w) = 4^2\, e^{\pi i k/2}\,(1-w)^{-k/2}\,(1-\bar w)^{k/2}\,(1-|w|^2)^{-s}\,. \tag{6.10}$$

Remarks.

(1) Any $f \in \mathcal{E}_{s,k}(\Omega \cap \mathfrak{H})$ is real-analytic, since Δ_k is an elliptic operator. It is far from sure that for $f \in \mathcal{E}_{s,k}(\Omega \cap \mathfrak{H})$ the function $\Phi_{s,k} f$ has a real-analytic continuation to Ω. The analyticity of the continuation is an additional requirement. It determines f uniquely on all open connected subsets of Ω that contain $\Omega \cap \mathfrak{H}$.

(2) An example is the function $z \mapsto y^s$, which is in $\mathcal{B}\big(\{z \in \mathbb{C} \;:\; \operatorname{Im} z > -1\}\big)$, where the restriction $\operatorname{Im} z > -1$ arises from the singularity of $\Phi_{s,k}$ at $-i$.

(3) Another example, defined on the region $|z| > 1$, is $f(z) = \operatorname{Im}(-1/z)^s$, which leads to $F(z) = (1 + i/z)^s\,(1 - i/\bar z)^s$.

Definition 6.4. The space of *analytic boundary germs* on an open set $I \subset \mathbb{P}^1_{\mathbb{R}}$ is

$$\mathcal{W}^{\omega}_{s,k}(I) = \varinjlim_{\Omega} \mathcal{B}_{s,k}(\Omega)\,, \tag{6.11}$$

where Ω runs over the open sets in $\mathbb{P}^1_{\mathbb{C}}$ that contain I.

For $I = \mathbb{P}^1_{\mathbb{R}}$ the elements of $\mathcal{W}^{\omega}_{s,k}(\mathbb{P}^1_{\mathbb{R}})$ are represented by real-analytic functions on an annulus $1 - \varepsilon < \left|\frac{z-i}{z+i}\right| < 1 + \varepsilon$ such that $\Delta_k f = s(1-s) f$ on $1 - \varepsilon < \left|\frac{z-i}{z+i}\right| < 1$.

The use of the direct limit in (6.11) implies that for representatives $f_1 \in \mathcal{B}_{s,k}(\Omega_1)$ and $f_2 \in \mathcal{B}_{s,k}(\Omega_2)$ of φ, the functions $F_1 = \Phi_{s,k} f_1$ and $F_2 = \Phi_{s,k} f_2$ have real-analytic extensions that coincide on $\Omega_1 \cap \Omega_2$. This implies that $I \mapsto \mathcal{W}^{\omega}_{s,k}(I)$ is a sheaf.

Definition 6.5. The *restriction morphism* $\operatorname{res}_{s,k} \colon \mathcal{W}^{\omega}_{s,k} \to \mathcal{V}^{\omega}_{s,k}$ is induced by assigning to $f \in \mathcal{B}_{s,k}(\Omega)$ the restriction of $F = \Phi_{s,k} f$ to $\Omega \cap \mathbb{P}^1_{\mathbb{R}}$.

For example the function $h(z) = y^s$ in $\mathcal{B}_{s,k}(\{z \in \mathbb{C} \;:\; \operatorname{Im} z > -1\})$ leads to the restriction $x \mapsto (x+i)^{s+k/2}(x-i)^{s-k/2}$, which is analytic on $\mathbb{R}$.

Lemma 6.6. *For each $g \in G$ the operators $|_k g \colon \mathcal{B}_{s,k}(\Omega) \to \mathcal{B}_{s,k}(g^{-1}\Omega)$ with $\Omega \supset I$ induce an operator $|_k g \colon \mathcal{W}^{\omega}_{s,k}(I) \to \mathcal{W}^{\omega}_{s,k}(g^{-1} I)$. Furthermore,*

$$(\operatorname{res}_{s,k}\varphi)|^{\mathrm{ps}}_{s,k} f = \operatorname{res}_{s,k}(f|_k g)\,. \tag{6.12}$$

We note that principal series action $|^{\mathrm{ps}}_{s,k} g$ on the sections $\mathrm{res}_{s,k}\varphi$ of $\mathcal{V}^{\omega}_{s,k}$ is related to the action $|_k g$ on boundary germs. The latter action does not depend on s.

Proof. The existence of the operators $|_k g$ follows from the direct limit definition of $\mathcal{W}^{\omega}_{s,k}(I)$.

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ near to $I_2 \in G$ a check of (6.12) is a long but straightforward computation. For the right hand side we know that the quantity $F_g(z)$ defined by

$$F_g(z) \;=\; \Phi_{s,k}(z)\, e^{-ik\arg(cz+d)} f(gz) \tag{6.13}$$

for $z \in g^{-1}\Omega \cap \mathfrak{H}$ extends to $g^{-1}\Omega$. For the left hand side we have

$$F(z) \;=\; \Phi_{k,s}(z)\, f(z) \tag{6.14}$$

on $\Omega \cap \mathfrak{H}$, and we know that it extends to Ω. We can eliminate f from the relation, and end up with a relation in terms of z and $\bar z$. Working out this relation takes some care with powers of complex quantities with complex exponents, but for $g \approx I_2$ this causes no problems. Then we substitute $z = t$ and $\bar z = t$ with $t \in g^{-1}I$, and observe that we get the factor in (3.1). The resulting relation extends as the equality of two multi-valued real-analytic functions on G. We have chosen the branches for $|_k g$ and $|^{\mathrm{ps}}_{s,k} g$ in the same way. □

The restriction morphism is not a morphism of G-equivariant sheaves. Tensoring with X_ρ we get a morphism of Γ-equivariant sheaves $\mathrm{res}_{s,k} : \mathcal{W}^{\omega}_{\rho v_k,s,k} \to \mathcal{V}^{\omega}_{\rho v_k,s,k}$.

Proposition 6.7 (Kernel functions $R_{s,k}$ and $q_{s,k}$)**.** *For $2s \not\equiv k \bmod 2$*

$$\begin{aligned} (\mathrm{res}_{s,k} q_{s,k}(z_1,\cdot))(t) \;&=\; b(s,k) R_{s,k}(t,z_1)\,, \\ b(s,k) \;&=\; e^{\pi i k/2}\, \frac{\Gamma(s-k/2)\,\Gamma(s+k/2)}{\Gamma(2s)}\,. \end{aligned} \tag{6.15}$$

Proof. In part (iv) of Proposition 6.2 the kernel function $q_{s,k}$ transforms with weight $-k$ in z_1 and with weight k in z_2. The Poisson kernel $R_{s,k}(t,z)$ transforms with weight $-k$ in z, and with a principal series action of weight k in t; see (4.3). This shows that it is sensible to compare the functions $z_2 \mapsto q_{s,k}(z_1,z_2)$ and $t \mapsto R_{s,k}(t,z_1)$.

The transformation behavior of both kernels implies that it suffices to take $z_1 = i$. We denote the gamma factors in (6.1) by

$$\mathrm{Gf} \;=\; \frac{\Gamma(s-k/2)\,\Gamma(s+k/2)}{\Gamma(2s)}\,, \tag{6.16}$$

and obtain with (6.1)

$$\begin{aligned} q_{s,k}(i,z) \;&=\; (1+i\bar z)^{k/2}\,(1-iz)^{-k/2}\,\mathrm{Gf}\left(\frac{y}{|z+i|^2}\right)^{s}\, {}_2F_1\Big(\frac{y}{|z+i|^2}\Big)\,, \\ \Phi_{s,k}(z) \;&=\; y^{-s}(z+i)^{s+k/2}\,(\bar z - i)^{s-k/2}\,, \\ F(z) \;&=\; e^{\pi i k/2}\,\mathrm{Gf}\; {}_2F_1\Big(\frac{y}{|z+i|^2}\Big)\,. \end{aligned}$$

The hypergeometric factor equals 1 for $y = 0$, hence we get $F(x) = e^{\pi ik/2}\,\mathrm{Gf}$ not depending on $x \in \mathbb{R}$, and then also for $x = \infty$ by analytic continuation. Thus,

$$(\mathrm{res}_{s,k} q_{s,k}(i,\cdot))(t) \;=\; e^{\pi ik/2}\,\mathrm{Gf}\,. \tag{6.17}$$

We observe in (4.1) that $R_{s,k}(t,i) = 1$, which completes the proof. □

Theorem 6.8. *Let $s \in \mathbb{C} \smallsetminus \mathbb{Z}_{\le 0}$, and $2s \neq \pm k$. The restriction morphism*

$$\mathrm{res}_{s,k} : \mathcal{W}^{\omega}_{s,k} \to \mathcal{V}^{\omega}_{s,k}$$

is bijective and $(\mathrm{res}_{s,k} f)|^{\mathrm{ps}}_{s,k} g = \mathrm{res}_{s,k}(f|_k g)$ for all $g \in G$ for representatives f of sections of $\mathcal{W}^{\omega}_{s,k}$.

If $\mathrm{res}_{s,k}\varphi \in \mathcal{W}^{\omega}_{s,k}(I)$ extends holomorphically to a convex neighborhood Ω in $\mathbb{P}^1_{\mathbb{C}}$ of the open interval $I \subset \mathbb{P}^1_{\mathbb{R}}$ not containing i and $-i$ and symmetric under complex conjugation, then φ can be represented by $f \in \mathcal{B}_{s,k}(\Omega)$ for the same neighborhood Ω.

Proof. Lemma (6.6) gives the intertwining property of the operators $|_k g$. Hence we can work with sections over an interval I contained in $\mathbb{R}$. Let $f \in \mathcal{B}_{s,k}(\Omega)$ represent a section $\varphi \in \mathcal{W}^{\omega}_{s,k}(I)$. Near I we have

$$\Phi_{s,k}(x+iy) \;=\; y^{-s}\,\alpha(x,y)\,, \tag{6.18}$$

with $\alpha(x,y) = (x+iy+i)^{s+k/2}\,(x-iy-i)^{s-k/2}$. Since the factor α is real-analytic without zeros on the strip $|y| < 1$ in $\mathfrak{H}$, the function

$$H(x,y) = y^{-s}\,f(x,y) \;=\; F(x,y)/\alpha(x,y) \tag{6.19}$$

is also real-analytic on Ω, and we can work with H instead of F.

For the injectivity we suppose that $\varphi = 0$ on I, and have to show that then $H = 0$ on a neighborhood of I in $\mathbb{C}$. The differential equation $\Delta_k f = s(1-s)f$ implies that H satisfies

$$-y^2\Big(\partial_x^2 H + \partial_y^2 H\Big) - 2sy\,\partial_y H + iky\,\partial_x H \;=\; 0\,. \tag{6.20}$$

Since H is real-analytic there is for each $x \in I$ an expansion $\sum_{n\ge0} a_n(x) y^n$ converging to $H(x,y)$ for y in an open interval containing 0. (This interval may depend on x.) Inserting this into the differential equation we get

$$a_n(x) \;=\; \begin{cases} \frac{ik}{2s} a_0'(x) & \text{if } n = 1\,,\\ \dfrac{ika_{n-1}'(x) - a_{n-2}''(x)}{n(2s+n-1)} & \text{if } n \ge 2\,. \end{cases} \tag{6.21}$$

(We use that $s \notin \mathbb{Z}_{\le 0}$.)

Since $a_0(x) = \varphi(x)$, the function a_n can be written as

$$a_n(x) \;=\; p_n\,\varphi^{(n)}(x)\,, \tag{6.22}$$

with coefficients depending on s and k. If $\varphi = 0$, then H vanishes on a neighborhood of I. Hence the restriction map is injective. (In the case $k = 0$ there is a nice formula for the a_n in [6, (5.15)]. We did not try to find a similar formula for general real weights.)

In [6, §5.2] the surjectivity of the restriction is shown in two ways: With a power series expansion (Theorem 5.6 in [6]) and with an integral representation (Theorem 5.7 in [6]). Here we try to generalize the latter approach.

Let Ω be a neighborhood of I with the properties indicated in the theorem. For given $\varphi \in \mathcal{V}^{\omega}_{\rho v_k,s,k}(I)$ extending holomorphically to Ω we put

$$f(z) \;=\; \frac{1}{i2^{2s-1}b(s,k)} \int_{t=\bar z}^{z} R_{1-s,-k}(t;z)\,\varphi(t)\,\frac{dt}{1+t^2}\,, \tag{6.23}$$

initially for $\operatorname{Re} s > \frac{1}{2}|k|$. The corresponding function F in (6.9) is

$$\begin{aligned} F(z) \;=\; \Phi_{k,s}(z) f(z) \;=\; & \frac{1}{i2^{2s-1}b(s,k)} \int_{t=\bar z}^{z} (z+i)^{s+k/2}\,(\bar z - i)^{s-k/2} \\ & \cdot \Bigl(\frac{t-i}{t-z}\Bigr)^{1-s+k/2} \Bigl(\frac{t+i}{t-\bar z}\Bigr)^{1-s-k/2} \varphi(t)\,\frac{dt}{t^2+1} \\ =\; & \frac{1}{i2^{2s-1}b(s,k)} \int_{t=-i}^{i} (x+iy+i)^{s+k/2}\,(x-iy-i)^{s-k/2} \\ & \cdot \Bigl(\frac{ty+x-i}{t-i}\Bigr)^{-s+k/2} \Bigl(\frac{ty+x+i}{t+i}\Bigr)^{-s-k/2} \varphi(x+yt)\,\frac{dt}{1+t^2}\,. \end{aligned}$$

It is clear that the integral converges absolutely for $\operatorname{Re} s$ sufficiently large, and that it describes a real-analytic function in $z = x+iy$. We take the value at $y=0$:

$$\begin{aligned} F(x) \;=\; & \frac{1}{i2^{2s-1}b(s,k)} (x+i)^{s+k/2}\,(x-i)^{s-k/2} \varphi(x) \\ & \cdot \int_{t=-i}^{i} \Bigl(\frac{x-i}{t-i}\Bigr)^{-s+k/2} \Bigl(\frac{x+i}{t+i}\Bigr)^{-s-k/2} \frac{dt}{1+t^2} \;=\; \varphi(x)\,. \end{aligned} \tag{6.24}$$

Under the assumption that $\operatorname{Re} s$ is large, this shows that φ occurs as the restriction of f. That is the surjectivity of $\operatorname{res}_{s,k}$. Moreover, if φ is holomorphic on a set Ω as indicated in the theorem, then F is real-analytic on Ω, and furthermore $f \in \mathcal{E}_{s,k}(\Omega \cap \mathfrak{H})$.

The integral

$$\int_{t=-i}^{i} \Bigl(\frac{ty+x-i}{t-i}\Bigr)^{-s+k/2} \Bigl(\frac{ty+x+i}{t+i}\Bigr)^{-s-k/2} \varphi(x+yt)\,\frac{dt}{1+t^2} \tag{6.25}$$

is holomorphic in $(s,k) \in \mathbb{C}^2$ on the region $\operatorname{Re} s > |\operatorname{Re} k|/2$. We aim at a meromorphic continuation for $(s,k) \in \mathbb{C}^2$. As t runs from $-i$ to i the term $ty+x$ runs from $t_- = \bar z$ to $t_+ = z$. The factor $\bigl(\frac{ty+x-i}{t-i}\bigr)^{-s+k/2}$ is holomorphic on $\mathbb{P}^1_{\mathbb{C}}$ except for a path from $t_+ = z$ to i, which we choose as indicated in Figure 2. The other factor is well-defined outside the path from $t_- = \bar z$ to $-i$ in the figure.

We replace the integration over $[-i,i]$ in (6.25) by integration over the Pochhammer contour P sketched in Figure 2. For Ω with the properties mentioned in the theorem we can arrange that the contour P is contained in Ω. The paths from $t_\pm$ to $\pm i$ are important only on the contour. A given choice of $p_\pm$ can be used for z varying over compact sets.

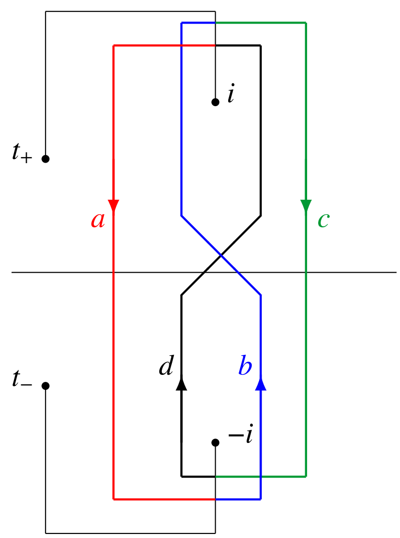


FIGURE 2. Pochhammer contour

We conclude that

$$\int_{t\in P} \Big(\frac{ty+x-i}{t-i}\Big)^{-s+k/2} \Big(\frac{ty+x+i}{t+i}\Big)^{-s-k/2} \varphi(x+yt)\, \frac{dt}{1+t^2} \tag{6.26}$$

depends analytically on $(s,k,z)\in\mathbb{C}^2\times\Omega$, holomorphically depending on (s,k).

To relate the outcome of (6.26) to the outcome of (6.25) we take $\operatorname{Re} s > \frac{1}{2}|\operatorname{Re} k|$. Then we can compute the integral over the Pochhammer contour as a linear combination of four integrals from $-i$ to i. We take the arguments in such a way that at $t=0$ on part a the argument of $\frac{ty+x-i}{t-i} = \arg(1+ix)\in(0,\pi)$, and the argument of $\frac{ty+x+i}{t+i}$ is equal to $\arg(1-ix)\in(-\pi,0)$. That is the choice of the arguments that we use in the computation of (6.24). In this way the transition from (6.25) to (6.26) amounts to multiplication by

$$-1+e^{-2\pi is-\pi ik}-e^{-2\pi ik}+e^{2\pi is-\pi ik} \;=\; 4e^{-\pi ik/2}\,\sin\Big(\pi\frac{k}{2}-\pi s\Big)\,\sin\Big(\pi\frac{k}{2}+\pi s\Big).$$

Hence f and F have a meromorphic extension in (s,k) with singularities occurring only in $s=\pm\frac{k}{2}$. □

6.4. **Restriction and one-sided averages.** Let $\beta<\alpha$. Based on a fixed function $\varphi\in C^\omega_\rho((\alpha,\beta)_c)$ we have the meromorphic family

$$s\mapsto f_s(z)=\operatorname{res}^{-1}_{s,k}\varphi\,(z) \;=\; \frac{1}{b(s,k)}\int_{t=\bar z}^{z} R_{1-s,-k}(t,z)\,\varphi(t)\frac{dt}{1+t^2}\,.$$

Like in [7, Lemma 4.6] we can try to get the lower row of the following scheme:

$$\begin{array}{ccc} \varphi & \xrightarrow{\mathrm{Av}^+} & \varphi|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^+ \\ {\scriptstyle \mathrm{res}^{-1}_{s,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}^{-1}_{s,k}} \\ f_s & \xrightarrow{\mathrm{Av}^+ ?} & \mathrm{res}^{-1}_{s,k}(\varphi|^{\mathrm{ps}}_{\rho v_k,s,k}\mathrm{Av}^+) \end{array} \tag{6.27}$$

We formulate the result that we will use later on.

Lemma 6.9. *Let* $\mathrm{Re}\, s > 0$*, and denote*

$$\begin{aligned} \Omega_\pm &= \mathbb{P}^1_{\mathbb{C}} \smallsetminus \left\{ z \in \mathbb{C} \ : \ \left| z \pm \frac{1}{2} \right| \le \frac{1}{2} \right\}, \\ \Upsilon_\pm &= \{ z \in \mathbb{C} \ : \ \pm \mathrm{Re}\, z > 0 \}. \end{aligned}$$

See Figure 3.

Suppose that $h_\pm \in X_\rho \otimes \mathcal{B}_{s,k}(\Omega_\pm)$ *and that the associated real-analytic functions* $H_\pm = \Phi_{s,k} h_\pm$ *on* $\Omega_\pm$ *satisfy* $\left(F_\pm(\infty), e_l\right)_\rho = 0$ *for all basis elements* e_l *with* $\kappa_l = 0$*. (See §2.2.) Then*

$$f_\pm|_{\rho v_k,k}\mathrm{Av}^\pm = \begin{cases} \sum_{m \ge 0} f_+|_{\rho v_k,k} T^m \\ -\sum_{m \le -1} f_-|_{\rho v_k,k} T^m \end{cases} \tag{6.28}$$

are well-defined elements of $X_\rho \otimes B_{s,l}(\Upsilon_\pm)$ *which satisfy*

$$\mathrm{res}_{s,k}\big(f_\pm|_{\rho v_k,k}\mathrm{Av}^\pm\big) = \big(\mathrm{res}_{s,k} f_\pm\big)\,\Big|^{\mathrm{ps}}_{\rho v_k,s,k}\ \mathrm{Av}^\pm\,. \tag{6.29}$$

Moreover, for $l = 1, \ldots, n(\rho)$*:*

$$\left(f_\pm|_{\rho v_k,k}\mathrm{Av}^\pm(x+iy), e_l\right)_\rho = \mathrm{O}(y^{-s}) \tag{6.30}$$

as $y \uparrow \infty$*, uniform for* x *in compact sets in* $\Upsilon_\pm \cap \mathbb{R}$*.*

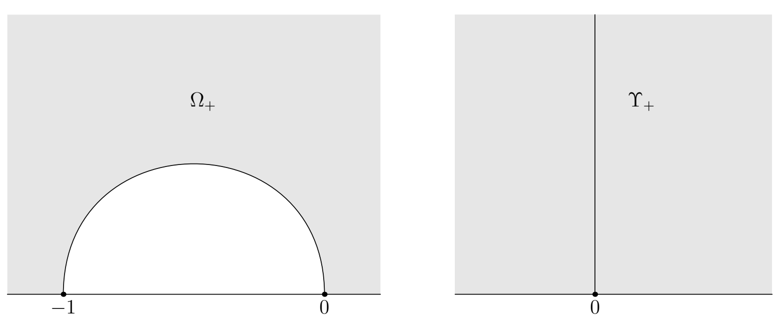


FIGURE 3. Domains Ω_+ and Υ_+ in Lemma 6.9.

Proof. The definition of $|_{\rho v_k,k}\mathrm{Av}^\pm$ follows the same scheme as the definition of $|^{\mathrm{ps}}_{\rho v_k,s,k}$. Only the power series $F_\pm$ at ∞ is in the variables z and $\bar{z}$. In the region of convergence the correspondence (6.29) is clear. It extends by analytic continuation in s.

In (6.30) we are interested in the asymptotic behavior as $y = \operatorname{Im} z \uparrow \infty$, whereas part (iii) of Proposition 5.4 concerns expansions for t approaching ∞ through $\mathbb{R}$.

It suffices to consider the component $f_{\pm,l}(z) = \big(f_\pm(z), e_l\big)_\rho$, and the corresponding component $F_{\pm,l} = \Phi_{s,k}\, f_{\pm,l}$. The function $F_{\pm,l}$ is given by a power series in $(z+i)^{-1}$ and $(\bar z^{-1} - i)^{-1}$ that converges on a neighborhood of ∞ in $\mathbb{P}^1_{\mathbb{C}}$.

If $F_{\pm,l}(\infty) = 0$ we have

$$F_{\pm,l}(z) \;=\; \Phi_{s,k}(z) f_{\pm,l}(z) \;=\; \mathrm{O}\big(y^s((y+1)^2 + x^2)^{-s-1/2}\big)\,.$$

This gives in the + case for $x \geq 0$ and $y \uparrow \infty$

$$\begin{aligned}\sum_{m\geq 0} e^{-2\pi i m\kappa_l}\, f_{\pm,l}(z+m) \;&\ll\; y^s \sum_{m\geq 0} (y^2 + (x+m)^2)^{-s-1/2}\\ &\ll\; y^{-s} \int_{t=x}^{\infty} (1+t^2)^{-s-1/2} \;=\; \mathrm{O}(y^{-s})\,.\end{aligned}$$

For the − case, replace $\sum_{m\geq 0}$ by $-\sum_{m\leq -1}$.

We treat the constant term of $F_{\pm,l}$ at ∞ separately. This needs to be done only for $\kappa_l \neq 0$.

$$\begin{aligned}\sum_{m=0}^{\infty} e^{2\pi i m\kappa_l}\, \Phi_{s,k}(z+m)^{-1} \;&=\; \sum_{m\geq 0} e^{-2\pi i m\kappa_l}\, y^s\, (z+m+i)^{-s-k/2}\, (\bar z + m - i)^{-s+k/2}\\ &=\; y^s \sum_{m\geq 0} e^{-2\pi i m\kappa_l}\, (z+m)^{-2s}\Big(1 + \mathrm{O}\big((z+m)^{-1}\big)\Big)\,.\end{aligned}$$

The O-term gives a convergent sum for $\operatorname{Re} s > 0$ with the estimate $\mathrm{O}(y^{-s})$ like above. The main term is y^s times the Lerch transcendent $H(2s, e^{-2\pi i\kappa_l}, z)$; see (5.1). We apply the asymptotic behavior in Proposition 5.2(ii) to $z = x + iy$ with x in a compact set, and $y \uparrow \infty$. Since $\zeta = e^{-2\pi i\kappa_l} \neq 1$, the expansion starts with z^{-2s}. This gives the desired result.

The individual terms in the sum (6.28) correspond to the individual terms with $\operatorname{res}_{s,k} f$. In the region of absolute convergence the relation in (6.29) is clear. This relation extends analytically in s. □

7. From period functions to cuspidal Maass forms

In Sections 2–4 we carried out the following steps

$$\begin{array}{ccccccc} u & \mapsto & \eta_{s,k}(u) & \mapsto & c^u_{\mathrm{par}} & \mapsto & c^u_{\mathrm{par}}(0,\infty)|_{(0,\infty)}\\ \in \mathcal{A}^0_k(s,\rho v_k) & & \text{see (4.11)} & & (\mathbb{P}^1_{\mathbb{Q}})^2 \to \mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}}) & & \in \mathsf{FE}^\omega_{\rho v_k,s,k}\end{array}$$

In this section the aim is to go from a period function to an automorphic form, using cocycles with values in the boundary germs, instead of in analytic functions on intervals in $\mathbb{P}^1_{\mathbb{R}}$. We state the main theorem.

Theorem 7.1. *Let $k \in \mathbb{R}$ and $s \in \mathbb{C}$ such that $\operatorname{Re} s \in (0,1)$ and $s \not\equiv \pm k/2 \bmod 1$. Let ρ be a finite-dimensional unitary representation of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then the linear map $\mathcal{A}^0_k(s,\rho v_k) \to \mathsf{FE}^\omega_{\rho v_k,s,k}$ given by $u \mapsto c^u_{\mathrm{par}}(0,\infty)|_{(0,\infty)}$ with $c^u_{\mathrm{par}}(\cdot,\cdot)$ as in (4.15) is bijective.*

We emphasize that the proof of Theorem 7.1 provides the inverse map. However, the statement of the inverse map is rather involved; it involves a transition to boundary germs. The proof is split into a number of steps. At the end of this section we give a recapitulation. In this section we use s, k and ρ as indicated in the theorem.

Use of the Farey tesselation. We use the Farey tesselation Ft illustrated in Figure 1. By X_0^{Ft} we denote the set of vertices, by X_1^{Ft} the set of edges $e_{\xi,\eta}$ in the tesselation, and by X_2^{Ft} the set of cells. The group $\bar\Gamma = \Gamma/\{\pm I_2\}$ acts on these sets, and $X_0^{\mathsf{Ft}} = \Gamma\,\infty$, $X_1^{\mathsf{Ft}} = \Gamma\, e_{0,\infty}$, and $X_2^{\mathsf{Ft}} = \Gamma C_{0,\infty,-1}$, where $C_{0,\infty,-1}$ denotes the cell with vertices 0, ∞ and -1. For each $e \in X_1^{\mathsf{Ft}}$ we choose an orientation, and use $e_{\eta,\xi} = -e_{\xi,\eta}$ to handle the opposite orientation. Since $S\, e_{0,\infty} = e_{\infty,0} = -e_{0,\infty}$, all oriented edges can be written as $\gamma^{-1} e_{0,\infty}$ with a unique $\gamma \in \bar\Gamma$.

Like in [7, §11.1, 11.3] the complex $F_\bullet^{\mathsf{Ft}} = \mathbb{C}[X_\bullet^{\mathsf{Ft}}]$ forms a resolution of $\bar\Gamma$-modules that leads to the parabolic cohomology groups $H^j(F_\bullet^{\mathsf{Ft}}; M) \cong H^j_{\mathrm{par}}(\bar\Gamma; M)$, $j = 0, 1, 2$, of $\bar\Gamma$-modules M. (Here we do not need the mixed cohomology groups used in [7].) Furthermore, $H^j_{\mathrm{par}}(\bar\Gamma; M) = H^j_{\mathrm{par}}(\Gamma; M)$ for modules in which the action of $-I_2$ is trivial.)

Lemma 7.2. *Let $\mathcal{W}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ correspond to $\mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ under the isomorphism $\mathrm{res}_{s,k}$ in Theorem 6.8. There is an injective linear map*

$$\beta_{s,k} \colon \mathsf{FE}^\omega_{\rho v_k,s,k} \to Z^1(F_\bullet^{\mathsf{Ft}}; \mathcal{W}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}}))\,.$$

Proof. A period function $f \in \mathsf{FE}^\omega_{\rho v_k,s,k}$ is a real-analytic function on $(0,\infty)$. The definition in (3.1) implies that

$$\tilde f(t) = \begin{cases} f(t) & \text{for } t \in [0,\infty]\,,\\ -f|^{\mathrm{ps}}_{\rho v_k,s,k} S\,(t) & \text{for } t \in [\infty,0]_c \end{cases} \tag{7.1}$$

is a continuous function on $\mathbb{P}^1_{\mathbb{R}}$ that has values in the 1-eigenspace of $|^{\mathrm{ps}}_{\rho v_k,s,k}(-I_2)$ and satisfies $\tilde f|^{\mathrm{ps}}_{\rho v_k,s,k} S = -\tilde f$. We can check that $\tilde f = \tilde f|^{\mathrm{ps}}_{\rho v_k,s,k}(T + T')$ on $\mathbb{P}^1_{\mathbb{R}}$. We determine $c_{\mathsf{Ft}} \in Z^1(F_\bullet^{\mathsf{Ft}}; \mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}}))$ by $c_{\mathsf{Ft}}(e_{0,\infty}) = \tilde f$, and extending this by $c_{\mathsf{Ft}}(\gamma^{-1} e_{0,\infty}) = \tilde f|^{\mathrm{ps}}_{\rho v_k,s,k}\gamma$. To see that c_{Ft} is a cocycle it suffices to show that $dc_{\mathsf{Ft}}(C_{0,\infty,-1}) = 0$. That is just the relation $\tilde f = \tilde f|^{\mathrm{ps}}_{\rho v_k,s,k}(T + T')$.

Since $\mathrm{res}_{s,k}$ is an isomorphism of Γ-equivariant sheaves (Theorem 6.8) there is a cocycle

$$b_{\mathsf{Ft}} = \mathrm{res}^{-1}_{s,k} c_{\mathsf{Ft}} \tag{7.2}$$

in $Z^1(F_\bullet^{\mathsf{Ft}}; \mathcal{W}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}}))$. It is the zero cocycle only if $\tilde f = 0$, and hence $f = 0$. So taking $\beta_{s,k} f = b_{\mathsf{Ft}}$ gives an injective linear map. □

The boundary germs in $\mathcal{W}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ are represented by elements of $\mathcal{B}_{s,k}(\Omega)$ where $\Omega \subset \mathbb{P}^1_{\mathbb{C}}$ is a neighborhood of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus E$, for a finite set E of cusps. We define a module of functions on $\mathfrak{H}$ containing a special choice of these representatives.

By $\mathcal{E}_{\rho v_k,s,k}(U)$ we denote $X_\rho \otimes E_{s,k}(U)$ with the action $|_{\rho v_k,k}$ of Γ.

Definition 7.3. Let $\mathcal{G}^{\omega^0,0}_{\rho v_k,s,k}$ be the space of functions $f \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H} \smallsetminus E_1)$ for a finite set $E_1 \subset X_1^{\mathsf{Ft}}$ of edges of the Farey tesselation; this set may depend on f. The finitely many connected components C of $\mathfrak{H} \smallsetminus E_1$ can be of the following types:

(1) The closure $\bar{C}$ of C in $\mathbb{P}^1_{\mathbb{C}}$ has finite area. In this case we require that the restriction f_C is in $\mathcal{E}_{\rho v_k,s,k}(C) = X_\rho \otimes \mathcal{B}_{s,k}(C)$.

(2) The closure $\bar{C}$ of C in $\mathbb{P}^1_{\mathbb{C}}$ contains one or more intervals I_j in $\mathbb{P}^1_{\mathbb{R}}$. We require that $f_C \in X_\rho \otimes \mathcal{B}_{s,k}(\Omega)$ for an open set $\Omega \subset \mathbb{P}^1_{\mathbb{C}}$ containing C and the intervals I_j.

Let $E \subset \mathbb{P}^1_{\mathbb{Q}}$ be the finite set of endpoints of the geodesics in E_1. The functions $\mathrm{res}_{s,k}\Phi_{s,k} f_C$ determine an element $\varphi \in \mathcal{V}^{\omega}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}} \smallsetminus E)$. We require that this element extends continuously to $\mathbb{P}^1_{\mathbb{R}}$.

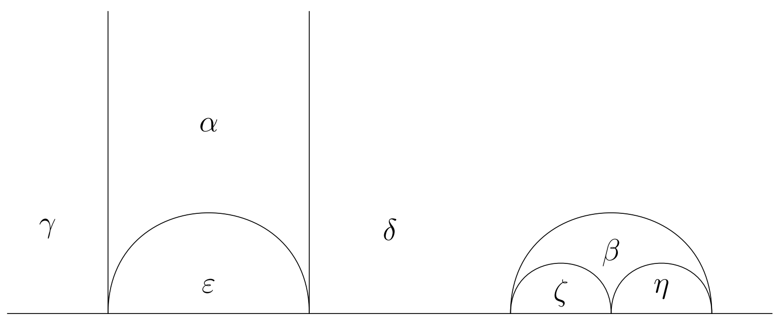


FIGURE 4. Decomposition of $\mathfrak{H}$ by a finite number of edges in the Farey tesselation.

In Figure 4 the components indicated by α and β are of type (1), the other components are of type (2). The closure of the component indicated by δ contains two intervals in $\mathbb{P}^1_{\mathbb{R}}$.

We note that $\mathcal{G}^{\omega^0,0}_{\rho v_k,s,k}$ is invariant under the action $|_{\rho v_k,k}$ of Γ. This Γ-module is not equal to the module $\mathcal{G}_s^{\omega^*,\mathrm{exc}}$ in [7, Definition 9.21], but we use it in a similar way.

Lemma 7.4. *There is a cochain* $\tilde{b}_{\mathsf{Ft}} \in C^1(F_\bullet^{\mathsf{Ft}}; \mathcal{G}^{\omega^0,0}_{\rho v_k,s,k})$ *such that* $\tilde{b}_{\mathsf{Ft}}(e)$ *represents* $b_{\mathsf{Ft}}(e) \in \mathcal{W}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ *for all* $e \in X_1^{\mathsf{Ft}}$. *Moreover,* $\tilde{b}_{\mathsf{Ft}}(e) \in \mathcal{B}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \tilde{e})$ *where* $\tilde{e}$ *is the closure of the union of* e *and its complex conjugate.*

Proof. $b_{\mathsf{Ft}}(e_{0,\infty})$ is related to $c_{\mathsf{Ft}}(e_{o,\infty})$ by $\mathrm{res}_{s,t} b_{\mathsf{Ft}}(e_{0,\infty}) = c_{\mathsf{Ft}}(e_{0,\infty})$, and $c_{\mathsf{Ft}}(e_{0,\infty}) = \tilde{f}$ is holomorphic on $\mathbb{C} \smallsetminus i\mathbb{R}$. Theorem 6.8 implies that $b_{\mathsf{Ft}}(e_{0,\infty})$ has a representative in $\mathcal{B}_{\rho v_k,s,k}(\mathbb{C} \smallsetminus i\mathbb{R})$. Take $\tilde{b}_{\mathsf{Ft}}(e_{0,\infty})$ as this representative, and extend the definition in a Γ-equivariant way. This results in $\tilde{b}_{\mathsf{Ft}} \in C^1(F_\bullet^{\mathsf{Ft}}; \mathcal{G}^{\omega^0,0}_{\rho v_k,s,k})$. □

The cochain $\tilde{b}_{\mathsf{Ft}}$ represents the cocycle b_{Ft}. So we have

$$d b_{\mathsf{Ft}}(C_{0,\infty,-1}) \;=\; b_{\mathsf{Ft}}(e_{0,\infty} - e_{-1,\infty} - e_{0,-1}) \;=\; 0\,,$$

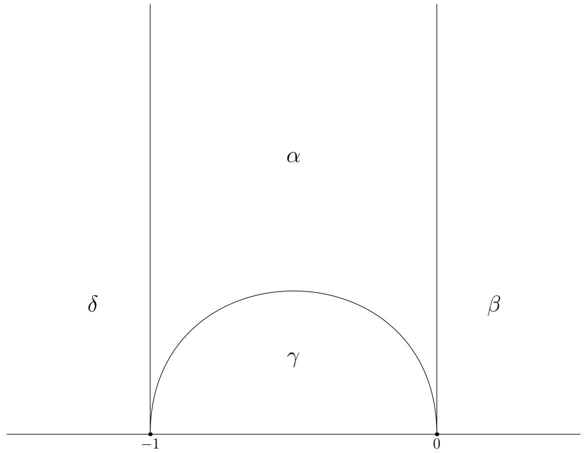


FIGURE 5. The cell $C_{0,\infty,-1}$ in the Farey tesselation.

which corresponds to the three term relation (3.6). In Figure 5 this means that $d\tilde{b}_{\mathsf{Ft}}(C_{0,\infty,-1})$ vanishes on the components β, γ and δ. On the component α the function $d\tilde{b}_{\mathsf{Ft}}(C_{0,\infty,-1})$ can be any $s(1-s)$-eigenfunction of Δ_k.

Lemma 7.5. *Given the cochain $\tilde{b}_{\mathsf{Ft}}$ representing the cocycle b_{Ft}, there exists a function $v \in \mathcal{E}_{\rho v_k,s,r}(\mathfrak{H})$ satisfying $v|_{\rho v_k,k}\gamma = v$ for all $\gamma \in \Gamma$. This establishes a linear map $\alpha_{s,k} \colon \mathsf{FE}^{\omega}_{\rho v_k,s,k} \to \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H})^{\Gamma}$.*

Proof. Let p be a positively oriented simple closed path along edges of the Farey tesselation Ft, as illustrated in Figure 6. Evaluating $\tilde{b}_{\mathsf{Ft}}(p)$ gives a function on $\mathfrak{H} \setminus p$. Since $\tilde{b}_{\mathsf{Ft}}$ represents the cocycle d_{Ft}, the function $\tilde{b}_{\mathsf{Ft}}(p)$ is equal to zero on the components outside p. On the open region $U(p) \subset \mathfrak{H}$ enclosed by p we obtain a function $v_p \in \mathcal{E}_{\rho v_k,s,k}(U(p))$, which depends on the path p.

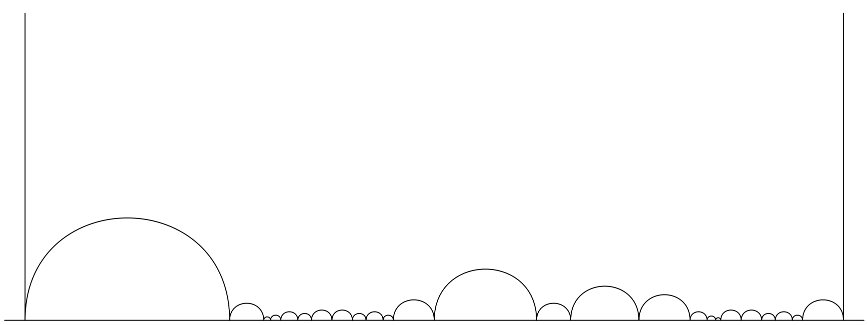


FIGURE 6. Closed path along edges of the Farey tesselation. (The boundary lines of the wide sector meet each other at ∞.)

If p_1 and p_2 are two paths for which $U(p_1) \cap U(p_2) \neq \emptyset$ we have $v_{p_1} = v_{p_2}$ on $U(p_1) \cap U(p_2)$. Indeed, the difference between p_1 and p_2 can be obtained by adding or subtracting successively a cell $\gamma^{-1}C_{0,\infty,-1}$ to or from the region. This changes only the value of $\tilde{d}_{\mathsf{Ft}}(p_1)$ on $\gamma^{-1}C_{0,\infty,-1}$, and $\gamma^{-1}C_{0,\infty,-1} \cap (U(p_1) \cap U(p_2)) = \emptyset$.

In particular, by making p wider and wider we obtain $v(p)$ on larger and larger regions. The limit as p tends to $\mathbb{P}^1_{\mathbb{R}}$ gives $v \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H})$.

Let $\gamma \in \Gamma$. For a given relatively compact open region $V \subset \mathfrak{H}$ we can take the path p encircling it sufficiently wide such that $\gamma^{-1}p$ encircles V as well. So on the region V we have

$$v(p) = \tilde{d}_{\mathsf{Ft}}(p) = \tilde{d}_{\mathsf{Ft}}(\gamma^{-1}p) = \tilde{d}_{\mathsf{Ft}}(p)|_{\rho v_k,k}\gamma .$$

So if $z, \gamma^{-1}z \in V$ then $v(p)(z) = v|_{\rho v_k,k}\gamma(z)$. In the limit this implies that $v \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H})^{\Gamma}$.

The map $\beta_{s,k}$ in Lemma 7.2, followed by $b_{\mathsf{Ft}} \mapsto \tilde{b}_{\mathsf{Ft}}$ is linear. Also the dependence of v on $\tilde{b}_{\mathsf{Ft}}$ is linear. The composition gives a linear map

$$\alpha_{s,k} \colon \mathsf{FE}^{\omega}_{\rho v_k,s,k} \to \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H})^{\Gamma} .$$

□

We now have the following situation:

$$\begin{array}{ccc} \mathsf{FE}^{\omega}_{\rho v_k,s,k} & \xrightarrow{\beta_{s,k}} & Z^1(F^{\mathsf{Ft}}_{\bullet}; \mathcal{W}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})) \\ & \searrow^{\alpha_{s,k}} & \downarrow \text{with Lemma 7.5} \\ & & \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H})^{\Gamma} \end{array} \tag{7.3}$$

In Proposition 4.2 we associated a period function to a Maass cusp form.

$$\begin{array}{cc} \mathsf{FE}^{\omega}_{\rho v_k,s,k} & \\ & \nwarrow^{\mathrm{pf}} \\ & \mathcal{A}^0_k(s, \rho v_k) \end{array} \tag{7.4}$$

The following lemma states that $\alpha_{s,k}$ is proportional to a left inverse of pf.

Lemma 7.6. *If $f \in \mathsf{FE}^{\omega}_{\rho v_k,s,k}$ is the period function of the Maass cusp form $u \in \mathcal{A}^0_k(s, \rho v_k)$, then*

$$\alpha_{s,k} f = \frac{2\pi i}{b(s,k)}\, u , \tag{7.5}$$

with the meromorphic factor $b(s,k)$ from (6.15).

Proof. Now $\tilde{f}$ in the proof of Lemma 7.2 is given by

$$c_{\mathsf{Ft}}(e_{0,\infty}) = \tilde{f} = \int_0^{\infty} \eta_{s,k}(u) = \int_0^{\infty} [u, R_{s,k}]_k , \tag{7.6}$$

and hence for each edge $e \in X^{\mathsf{Ft}}_1$

$$c_{\mathsf{Ft}}(e) = \int_e [u, R_{s,k}]_k . \tag{7.7}$$

See Proposition 4.1, (4.15) and (4.11). With Proposition 6.7 and the bijectivity of $\mathrm{res}_{s,k}$ in Theorem 6.8 we obtain

$$[u, R_{s,k}(z,\cdot)]_k = \frac{1}{b(s,k)}\,[u, q_{s,k}(z,\cdot)]_k\,, \tag{7.8}$$

and for $e \in X_1^{\mathsf{Ft}}$

$$\tilde b_{\mathsf{Ft}}(e) = \frac{1}{b(s,k)} \int_e [u, q_{s,k}(z,\cdot)]_k\,. \tag{7.9}$$

The exponential decay of u and its derivatives implies the absolute convergence of these integrals. We have

$$d\tilde b_{\mathsf{Ft}}(C_{0,\infty,-1}) = \tilde b_F(e_{0,\infty} + e_{\infty,-1} + e_{-1,0})\,. \tag{7.10}$$

Now we would like to apply Proposition 6.3. However the closed curve $\partial C_{0,\infty,-1}$ is not in contained in $\mathfrak{H}$. We can truncate the cell $C_{0,\infty,-1}$ at its vertices, and apply Proposition 6.3 to this approximation of $\partial C_{0,\infty,1}$. The exponential decay of u and its derivatives implies that the truncation error goes to zero in the limit. The result is

$$d\tilde b_{\mathsf{Ft}}(C_{0,\infty,-1}) = \frac{2\pi i}{b(s,k)}\,u \tag{7.11}$$

on the interior of $C_{0,\infty,-1}$. By analyticity this gives the lemma. □

Lemma 7.7. *The function $v = \alpha_{s,k} f$ associated to a period function $f \in \mathsf{FE}^{\omega}_{\rho v_k,s,k}$ is in $\mathcal{A}^0_k(s,\rho v_k)$.*

Proof. We have still to show that v has exponential decay. The equivariance of v implies that it suffices to give an estimate of $v(x+iy)$ as $y \uparrow \infty$ for x in an interval of length at least 1.

Let $f \in \mathsf{FE}^{\omega}_{\rho v_k,s_0,k}$ and denote by $\tilde b_{\mathsf{Ft}}$ the cochain in Lemma 7.4 with which we built $v \in \mathcal{E}_{\rho v_k,s_0,k}(\mathfrak{H})^{\Gamma}$ as in the proof of that lemma. Let $h = \tilde b_{\mathsf{Ft}}(e_{0,\infty})$. So $\mathrm{res}_{s,h} h = f$.

Proposition 5.6 implies that $f = (f|^{\mathrm{ps}}_{\rho v_k,s,k} T')|^{\mathrm{ps}}_{\rho v_k,s,k} \mathrm{Av}^+$. With (6.29) in Lemma 6.9 this implies that $h = (h|_{\rho v_k,k})|_{\rho v_k} \mathrm{Av}^+$. With (6.30) this implies that

$$h(z) = \mathrm{O}(y^{-s}) \qquad \text{as } y \uparrow \infty \tag{7.12}$$

uniform for x in compact sets contained in $(0,\infty)$. We use the closed path

$$p = -T^{-2} e_{0,\infty} - T^{-1} T'^{-1} e_{0,\infty} - T'^{-1} e_{0,\infty} + e_{0,\infty}\,,$$

sketched in Figure 7, encircling the union of two cells in the Farey tesselation. We aim to estimate $v(z)$ for z in the region bounded by dashed lines. For z inside the path p

$$\begin{aligned} v(z) &= -h|_{\rho v_k,k} T^2(z) - h|_{\rho v_k,k} T'T(z) - h|_{\rho v_k,k} T'(z) + h(z) \\ &= -\rho(T)^{-2} v_k(T)^{-2} h(z+2) - * e^{-ik\arg(z+2)}\, h\Big(\frac{z+1}{z+2}\Big) \\ &\quad - * e^{-ik\arg(z+1)}\, h\Big(\frac{z}{z+1}\Big) + h(z)\,. \end{aligned} \tag{7.13}$$

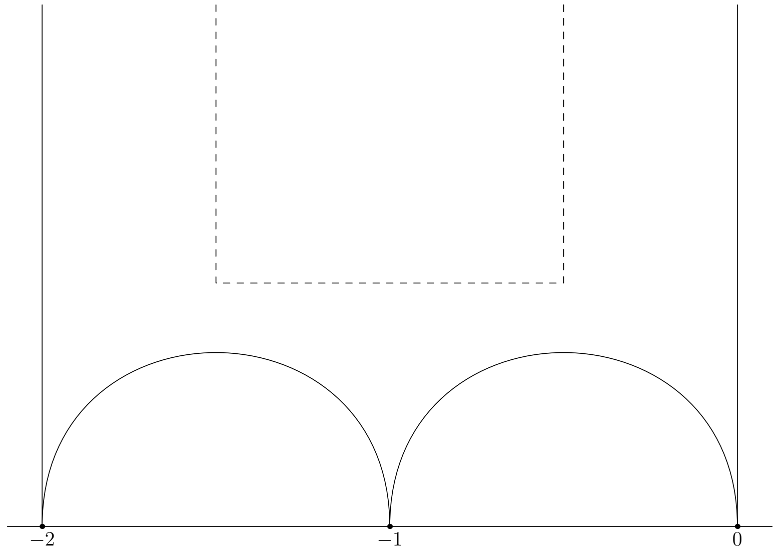


FIGURE 7. Closed path used in the proof of Lemma 7.7.

By $*$ we indicate quantities with absolute value 1 that do not depend on z.

For the first and the last term we have the estimate $\mathrm{O}(y^{-s})$; see (7.12). In the middle terms we have as $y \uparrow \infty$

$$\begin{aligned} h\Bigl(\frac{z+a}{z+a+1}\Bigr) &= \Phi_{s,k}\Bigl(\frac{z+a}{z+a+1}\Bigr)^{-1}\,\mathrm{O}(1) \\ &\ll \frac{y^s}{|z+a+1|^{2s}}\Bigl(\frac{z+a}{z+a+1}+i\Bigr)^{-s-k/2}\Bigl(\frac{\bar z+a}{\bar z+a+1}+i\Bigr)^{-s+k/2} \\ &= \mathrm{O}(y^{-s})\,. \end{aligned}$$

The conclusion is that $v(z) = \mathrm{O}(y^{-s})$ as $y \uparrow \infty$, first for $\frac{1}{2} \leq x \leq \frac{3}{2}$, and then for all x by T-equivariance.

For all components $v_l = (v, e_l)_\rho$ this implies that the exponentially increasing M-Whittaker functions in (2.13) do not occur in the Fourier expansion of v_l. For $\kappa_l \in (0,1)$ there are no Fourier terms with order n equal to zero. Then all Fourier terms are exponentially decreasing, and hence v_l is exponentially decreasing.

For components v_l with $\kappa_l = 0$ the Fourier term of order zero might contain a linear combination of y^s and y^{1-s} (or a logarithmic possibility if $s = \frac{1}{2}$). For $0 < \operatorname{Re} s < 1$ this is ruled out by the estimate $\mathrm{O}(y^{-s})$. (To obtain this we have used an assumption that holds in this case by Lemma 3.3.) □

We turn to the injectivity of the map $\alpha_{s,k}$ from period functions to Maass cusp forms.

Lemma 7.8. *For each* $f \in \mathsf{FE}^{\omega}_{s,k}$ *there is a holomorphic function* f_∞ *on* $\mathbb{C} \smallsetminus (i[1,\infty) \cup (-i)[1,\infty))$ *such that*

$$(7.14)\qquad \begin{aligned} f &= f_\infty - f_\infty|^{\mathrm{ps}}_{\rho v_k,s,k} S \qquad \text{on } \{z \in \mathbb{C} \,:\, \operatorname{Re} z > 0\}, \\ f_\infty &- f_\infty|^{\mathrm{ps}}_{\rho v_k,s,k} T \in \mathcal{V}^{\omega}_{s,k}(\mathbb{P}^1_{\mathbb{R}}). \end{aligned}$$

Proof. Let $\Omega_1 = \mathbb{C} \smallsetminus (i[1,\infty) \cup (-i)[a,\infty))$ and $\Omega_2 = \mathbb{P}^1_{\mathbb{C}} \smallsetminus i[-1,1]$. Then $\Omega_1 \cap \Omega_2 = \mathbb{C} \smallsetminus i\mathbb{R}$ and $\Omega_1 \cup \Omega_2 = \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i,-i\}$. We apply [14, Theorem 1.4.5] with $\Omega = \mathbb{P}^1_{\mathbb{C}}$ and follow the reasoning in the proof of [7, Proposition 13.1], obtaining from the holomorphic function f on $\Omega_1 \cap \Omega_2$, holomorphic functions A_∞ on Ω_1 and A_0 on Ω_2 such that $A_\infty + A_0 = f$ on $\Omega_1 \cap \Omega_2$. (Hörmander requires that we work with open subsets of $\mathbb{C}$. That is arranged by a holomorphic transformation of $\mathbb{P}^1_{\mathbb{C}}$ sending $-i$ to ∞.)

Note that A_0 is holomorphic on a neighborhood of ∞. Hence $A_\infty = f - A_0$ is in $\mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{R})$, and analogously $A_0 \in \mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\})$. In this way we can conclude that A_∞ and A_0 are elements of $\mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$.

We have

$$0 = f + f|^{\mathrm{ps}}_{\rho v_k,s,k} S = (A_\infty + A_0|^{\mathrm{ps}}_{\rho v_k,s,k} S) + (A_0 + A_\infty|^{\mathrm{ps}}_{\rho v_k,s,k} S).$$

Considering the singularities of the terms we conclude that

$$h = A_\infty + A_0|^{\mathrm{ps}}_{\rho v_k,s,k} S = -A_0 - A_\infty|^{\mathrm{ps}}_{\rho v_k,s,k} S$$

represents an element of $\mathcal{V}^{\omega}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$. We put

$$(7.15)\qquad \begin{aligned} f_\infty &= A_\infty - \frac{1}{2}h &&= \frac{1}{2}(A_\infty - A_0|^{\mathrm{ps}}_{\rho v_k,s,k} S), \\ f_0 &= A_0 + \frac{1}{2}h &&= \frac{1}{2}(A_0 - A_\infty|^{\mathrm{ps}}_{\rho v_k,s,k}). \end{aligned}$$

These functions satisfy $f_\infty + f_0 = f$ on $\mathbb{C} \smallsetminus i\mathbb{R}$, and $f_0|^{\mathrm{ps}}_{\rho v_k,s,k} S = -p_\infty$.

Since $A_\infty \in \mathcal{V}^{\omega}_{s,k}(\mathbb{R})$, we have $p_\infty|^{\mathrm{ps}}_{\rho v_k,s,k} T \in \mathcal{V}^{\omega}_{s,k}(\mathbb{R})$ as well. Analogously, $p_0 \in \mathcal{V}^{\omega}_{s,k}(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\})$. However,

$$f_\infty|^{\mathrm{ps}}_{s,k}(1-T) = p|^{\mathrm{ps}}_{s,k} T' - p_0|^{\mathrm{ps}}_{\rho v_k,s,k}(1-T),$$

which show that $f_\infty|^{\mathrm{ps}}_{s,k}(1-T) \in \mathcal{V}^{\omega}_{s,k}(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{-1,0\})$. Hence $f_\infty|^{\mathrm{ps}}_{s,k}(1-T) \in \mathcal{V}^{\omega}_{s,k}(\mathbb{P}^1_{\mathbb{R}})$. □

Lemma 7.9. *Let*

$$(7.16)\qquad \mathcal{F}_{s,k} = \lim_{\substack{\rightarrow \\ \Omega}} \mathcal{E}_{s,k}(\Omega \cap \mathfrak{H})$$

where Ω *runs over the open sets in* $\mathbb{P}^1_{\mathbb{C}}$ *that contain* $\mathbb{P}^1_{\mathbb{R}}$*. Then*

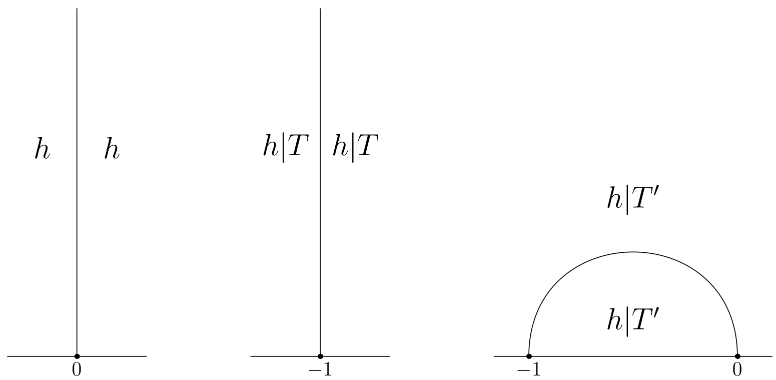
$$(7.17)\qquad \mathcal{F}_{s,k} = \mathcal{E}_{s,k}(\mathfrak{H}) \oplus \mathcal{W}^{\omega}_{s,k}(\mathbb{P}^1_{\mathbb{R}}).$$

Proof. This result generalizes [7, (3.3)]. In [7] the decomposition of boundary germs is based on Proposition 1.1, which we have generalized to weight k as Proposition 6.3. The reasoning leading to [7, (3.3)] generalizes as well. □

Lemma 7.10. *The map $\alpha_{s,k}$ in Lemma 7.5 is injective.*

Proof. Any period function $f \in \mathsf{FE}^{\omega}_{\rho v_k,s,k}$ determines a cocycle $\beta_{s,k}p$ on X_1^{Ft} with values in $\mathcal{W}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$ (Lemma 7.2), determined by its value on $e_{0,\infty}$. This cocycle is represented by a 1-cochain $\tilde{b}_{\mathsf{Ft}}$ on X_1^{Ft}, which is determined by $h = \tilde{b}_{\mathsf{Ft}}(e_{0,\infty}) \in \mathcal{G}^{\omega^0,0}_{\rho v_k,s,k}$. Lemma 7.4 implies that $H = \Phi_{s,k}\, h$ extends as a real-analytic function on $\mathbb{C} \setminus i\mathbb{R}$.

The function h is defined (and real-analytic) on $\mathfrak{H} \setminus e_{0,\infty}$. This implies that $h_{\rho v_k,k}T$ and $h_{\rho v_k,k}T'$ are defined (and real-analytic) on $\mathfrak{H} \setminus e_{-1,\infty}$ and $\mathfrak{H} \setminus e_{0,-1}$, respectively.



Hence, the function $l = h - h|_{\rho v_k,k}T - h|_{\rho,v_k,k}T'$ is in $\mathcal{E}_{s,k}(\mathfrak{H} \setminus U)$, where U is the union of the three geodesic boundary segments of the Farey cell. These geodesics determine four connected regions in the upper half-plane. The cochain $\tilde{b}_{\mathsf{Ft}}$ with values in $\mathcal{G}^{\omega^0,0}_{\rho v_k,s,k}$ represents the cocycle b_{Ft} with values in $\mathcal{W}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_{\mathbb{R}})$. Since b_{Ft} is a cocycle, the value of

$$\tilde{b}_{\mathsf{Ft}}(e_{0,\infty} - e_{0,-1} - e_{-1,\infty}) = l$$

should be zero near $\mathbb{P}^1_{\mathbb{R}} \setminus \{0, \infty, -1\}$. Since h is real-analytic on $\mathfrak{H} \setminus e_{0,\infty}$, the function l vanishes outside the triangle with vertices 0, ∞ and -1. On the other hand, inside the triangle, l represents (a multiple of) $\alpha_{s,k}f$, by the construction in the proof of Lemma 7.5.

Suppose now that $\alpha_{s,k}f = 0$. Then l is zero on all four components, and extends as the zero function on $\mathfrak{H}$. We note that $h \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H} \setminus e_{0,\infty})$, $h|_{\rho v_k,k}T \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H} \setminus e_{-1,\infty})$, and $h|_{\rho v_k,k}T' \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H} \setminus e_{-1,0})$. We have $h = h|_{\rho v_k,k}T + h|_{\rho,v_k,k}T'$. The right-hand side of this equality is in $\mathcal{E}_{\rho v_k,s,k}(U)$ for some open neighborhood U of $e_{0,\infty}$ in $\mathfrak{H}$, hence h is in $\mathcal{E}_{\rho v_k,s,k}(U)$. This implies that

$$h \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H})\,. \tag{7.18}$$

Theorem 6.8 states that the restriction $\mathrm{res}_{s,k} \colon \mathcal{W}^{\omega}_{s,k} \to \mathcal{V}^{\omega}_{s,k}$ is bijective. Moreover, it states that the domain of representatives is preserved. Lemma 7.8 splits $f \in \mathsf{FE}^{\omega}_{s,k}$ as $f = f_\infty - f_\infty|^{\mathrm{ps}}_{\rho v_k,s,k}S$. This implies that $h = h_\infty - h_\infty|_{\rho v_k,k}S$ with $h_\infty \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H} \setminus i[1,\infty])$ and $h_\infty|_{\rho v_k,k}S \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H} \setminus i(0,1])$. From the fact that $h \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H})$ we conclude that $h_\infty \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H} \setminus \{i\})$. Lemma 7.8 implies that

$h_\infty|_{\rho v_k,k}(1-T) \in \mathcal{G}^{\omega^0,0}_{s,k}(\mathbb{P}^1_\mathbb{R})$ represents an element of $\mathcal{W}^\omega_{s,k}(\mathbb{P}^1_\mathbb{R})$. The element h_∞ itself corresponds to the function f_∞, which is holomorphic on a neighborhood of $\mathbb{R}$ in $\mathbb{C}$. Hence h_∞ represents an element of $\mathcal{W}^\omega_{s,k}(\mathbb{R})$.

Let $h_\infty = \sum_l h_{\infty,l}\, e_l$ as in (2.11), for an eigenbasis $\{e_l\}$ of X_ρ for $\rho(T)$, as in (2.11). The action $h_\infty \mapsto h_\infty|_{\rho v_k,k}$ corresponds to $h_{\infty,l} \mapsto e^{-2\pi i\kappa_l}\, h_{\infty,l}|_k T$.

The element $h_{\infty,l}$ represents an element of $\mathcal{F}_{s,k}$ in (7.16). So we can write $h_{\infty,l} = g_l + q_l$ with $g_l \in \mathcal{E}_{s,k}(\mathfrak{H})$ and $q_l \in \mathcal{G}^{\omega^0,0}_{s,k}(\mathbb{P}^1_\mathbb{R})$ representing an element of $\mathcal{W}^\omega_{s,k}(\mathbb{P}^1_\mathbb{R})$. Since the eigenvalue of $|_{\rho v_k,k}T$ on e_l is $e^{2\pi i\kappa_l}$ we have

$$h_{\infty,l}\,\Big|_k\,(1-e^{-2\pi i\kappa_l}T) \;=\; g_l\,\Big|_k\,(1-e^{-2\pi i\kappa_l}T) + q_l\,\Big|_k\,(1-e^{-2\pi i\kappa_l}T)\,. \tag{7.19}$$

This is the situation which is treated in [7, Lemma 9.23]. Hence $h_{\infty,l} \in \mathcal{E}_{s,k}(\mathfrak{H}\smallsetminus\{i\})$. Moreover, $h_{\infty,l}|_k(1-e^{-2\pi i\kappa_l}T)$ represents an element of $\mathcal{W}^\omega_{s,k}(\mathbb{P}^1_\mathbb{R})$. So the term with $g_l \in \mathcal{E}_{s,k}(\mathfrak{H})$ is zero by the decomposition (7.17), and

$$g_l|_k T \;=\; e^{2\pi i\kappa_l}\, g_l\,. \tag{7.20}$$

Combining the component functions to vector-valued functions we obtain

$$h_\infty \;=\; g+q\,, \qquad g|_{\rho v_k,k}T \;=\; g\,, \tag{7.21}$$

with $g \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H})$, and $q \in \mathcal{G}^{\omega^0,0}_{\rho v_k,s,k}(\mathbb{P}^1_\mathbb{R})$ representing an element of $\mathcal{W}^\omega_{\rho v_k,s,k}(\mathbb{P}^1_\mathbb{R})$.

Since $h_\infty \in \mathcal{E}_{\rho v_k,s,k}(\mathfrak{H}\smallsetminus\{i\}) \cap \mathcal{G}^{\omega^0,-0}_{\rho v_k,s,k}(\mathbb{R})$, the function g represents an element of $\mathcal{W}^\omega_{\rho v_k,s,k}(\mathbb{R})$. The invariance of g under $|_{\rho v_k,k}T$ implies that it has a Fourier expansion of the type discussed in §2.2. In this expansion the W-Whittaker functions do not have the right behavior near zero to give a contribution in $\mathcal{W}^\omega_{\rho v_k,s,k}(\mathbb{R})$. In the terms of non-zero order we are left with multiples of M-Whittaker functions, and in the term of order zero with multiples of $z \mapsto y^s$.

Since q represents an element of $\mathcal{W}^\omega_{\rho v_k,s,k}(\mathbb{P}^1_\mathbb{R})$, this implies that $g(z) = \mathrm{O}(y^{-s})$ as $y \uparrow \infty$. Each Fourier term inherits this estimate. The function $z \mapsto y^s$ and the M-Whittaker functions have larger growth, and hence occur with coefficient zero. So $g = 0$, and $h_\infty = q_\infty$.

We use $h = h|_{\rho v_k,k}(1-S) = q_\infty|_{\rho v_k,k}(1-S)$ to see that it represents an element of $\mathcal{W}^\omega_{\rho v_k,s,k}(\mathbb{P}^1_\mathbb{R})$. We combine this with (7.18) to get $h = 0$ with Lemma 7.9, and also $f = \mathrm{res}_{s,k} h = 0$. □

Recapitulation of the proof of Theorem 7.1. Lemma 7.5 gives the central step, in which an invariant eigenfunction is constructed on the basis of the cochain $\tilde{b}_{\mathsf{Ft}}$ representing the cocycle b_{Ft}. Here it is important that we work with a cocycle with values in the boundary germs. This allows a geometrical approach in the upper half-plane.

Before Lemma 7.5 we have to construct a cocycle from a given period function. It is easy to get a cocycle with values in the sheaf $\mathcal{V}^{\omega^0,0}_{\rho v_k,s,k}$ based on the principal series realized on the boundary $\mathbb{P}^1_\mathbb{R}$ of $\mathfrak{H}$. To go over to boundary germs we use Theorem 6.8.

After Lemma 7.5 we have to show that the resulting invariant eigenfunctions have the desired properties. Lemma 7.7 shows that the invariant eigenfunctions are indeed Maass cusp form. Lemma 7.6 shows if we apply the construction to the period function associated to a Maass cusp form we get back (a non-zero multiple of) this cusp form. The final steps, in Lemmas 7.8 and 7.10, show that a non-zero period function gives a non-zero Maass cusp form.

8. Jacobi Maass forms

Jacobi Maass form have been studied by Yang [36, 37], and by Pitale [27].

In this final section we extend the definition of Jacobi Maass forms of Pitale to real weights, and show that spaces of Jacobi Maass cusp forms are isomorphic to spaces of vector-valued Maass cusp forms to which we can apply Theorem 7.1.

8.1. Jacobi group and its covering group.

The Jacobi group $G^J = \mathsf{Hei} \rtimes G$ is the semidirect product of $G = \mathrm{SL}_2(\mathbb{R})$ and the Heisenberg group Hei. As a topological space, $\mathsf{Hei} \cong \mathbb{R}^3$:

$$\mathsf{Hei} = \{\mathrm{h}(x,y,r) \;:\; x,y,r \in \mathbb{R}\}\,; \tag{8.1}$$

it has the group operation

$$\mathrm{h}(x_1,y_1,r_1)\,\mathrm{h}(x_2,y_2,r_2) = \mathrm{h}(x_1+x_2, y_1+y_2, r_1+r_2+x_1y_2-x_2y_1)\,. \tag{8.2}$$

The semidirect product is given by the following right action of G on Hei.

$$g^{-1}\mathrm{h}(x,y,r)g = \mathrm{h}(ax+cy, bx+dy, r) \qquad \text{with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{8.3}$$

We note that $(ax+cy, bx+dy) = (x,y)\,g$.

One can embed G^J in $\mathrm{GL}_4(\mathbb{R})$. Berndt and Schmidt describe G^J by such an embedding, see [2, §1.1].

The universal covering group of the Jacobi group is obtained as

$$\tilde G^J = \mathsf{Hei} \rtimes \tilde G\,,$$

with the universal covering group $\tilde G$ in §2.3. The action of $\tilde G$ on Hei is given by projection to G in (2.15):

$$\tilde g^{-1} h \tilde g = (\mathrm{pr}\,\tilde g)^{-1} h (\mathrm{pr}\,\tilde g) \qquad h \in \mathsf{Hei},\ \tilde g \in \tilde G\,. \tag{8.4}$$

Since Hei is simply connected, it is its own universal covering group.

A function $F \in C^\infty(\tilde G^J)$ has *index* $m \in \mathbb{R}_{\neq 0}$ and *weight* $k \in \mathbb{R}$ if

$$F(\mathbf{g}\mathrm{h}(0,0,r)\tilde{\mathrm{k}}(\vartheta)) = e^{2\pi i m r + ik\vartheta}\, F(\mathbf{g}) \qquad \text{for } \mathbf{g} \in \tilde G^J,\ r,\vartheta \in \mathbb{R}\,. \tag{8.5}$$

It is essential to put $\tilde{\mathrm{k}}(\vartheta)$ on the right. The elements $\mathrm{h}(0,0,r)$ are central in $\tilde G^J$ and can be put where it suits us. The index and the weight are not changed under left translation $L(\mathbf{g}g)$ for $\mathbf{g} \in \tilde G$.

The Jacobi group acts on $\mathfrak{H} \times \mathbb{C}$ by

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z) &= \Bigl(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Bigr) && \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\,, \\ \mathrm{h}(x,y,r)\cdot(\tau,z) &= (\tau, z+x\tau+y) && \text{for } \mathrm{h}(x,y,r) \in \mathsf{Hei}\,. \end{aligned} \tag{8.6}$$

This induces an action of $\tilde G^J$ on $\mathfrak{H}\times\mathbb{C}$ by

$$h\tilde g\cdot(\tau,z) \;=\; h\,\mathrm{pr}(\tilde g)\cdot(\tau,z)\,. \tag{8.7}$$

In the previous sections we denote elements of the upper half-plane $\mathfrak{H}$ by z, in accordance we the usual practice in the study of Maass forms. Here we follow the convention to denote by τ the modular variable in $\mathfrak{H}$, and use $z\in\mathbb{C}$ as the name of the elliptic variable.

Like in (2.28), there is a map

$$\Psi_{k,m}\colon C^\infty(\mathfrak{H}\times\mathbb{C})\to C^\infty(\tilde G^J) \tag{8.8}$$

determined by the following relation between F and $f=\Psi_{k,m}F$:

$$f(\mathrm{h}(p,q,r)\tilde{\mathrm{p}}(\tau)\tilde{\mathrm{k}}(\vartheta)) \;=\; e^{ik\vartheta+2\pi imr+2\pi imp(p\tau+q)}\,F(\tau,p\tau+q)\,, \tag{8.9}$$

with inverse relation

$$F(\tau,z) \;=\; e^{-2\pi imz\,\mathrm{Im}\,(z)/\mathrm{Im}\,(\tau)}\,f\Big(\mathrm{h}\Big(\tfrac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau},z-\tfrac{\tau\,\mathrm{Im}\,z}{\mathrm{Im}\,\tau},0\Big)\tilde{\mathrm{p}}(\tau)\Big)\,. \tag{8.10}$$

The right representation of $\tilde G^J$ by left translation $L(\mathbf{g}_1)f\colon\mathbf{g}\mapsto f(\mathbf{g}_1\mathbf{g})$ corresponds under $\Psi_{k,m}$ to a right representation of $\tilde G$ on functions on $\mathfrak{H}\times\mathbb{C}$ determined by

$$\begin{aligned}(F|^J_{k,m}\ell(g))(\tau,z) \;&=\; e^{-ik\arg(c\tau+d)}\,e^{-2\pi imcz^2/(c\tau+d)}\\ &\quad\cdot F\big(g(\tau,z)\big) && \text{for } g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G\,,\\ (F|^J_{k,m}\zeta)(\tau,z) \;&=\; e^{\pi ikn}\,F(\tau,z) && \text{for } \zeta\;=\;\tilde{\mathrm{k}}(\pi n)\in\tilde Z\,,\\ (F|^J_{k,m}h)(\tau,z) \;&=\; e^{2\pi im\left(r+p^2\tau+2pz+pq\right)}\,F(h(\tau,z)) && \text{for } h=\mathrm{h}(p,q,r)\,.\end{aligned} \tag{8.11}$$

For $k\in\mathbb{Z}$ this is a representation of $\tilde G^J$ which is trivial on $\tilde Z_2$, defined in (2.16). Hence it is a representation of G^J. It is the action used by Pitale, [27, (4)]. For $k\in\mathbb{R}\smallsetminus\mathbb{Z}$ the representation $|_{k,m}$ is not trivial on $\tilde Z_2$. The operators $|_{k,m}\ell(g)$ for $g\in G$ are similar to the operators $|_k g$ in (2.3).

8.2. **Discrete subgroup.** We use the discrete subgroup $\tilde\Gamma^J=\Lambda\rtimes\tilde\Gamma$ of $\tilde G^J=\mathsf{Hei}\rtimes\tilde G$, with the lattice

$$\Lambda \;=\; \mathsf{Hei}(\mathbb{Z}) \;=\; \{\mathrm{h}(\lambda,\mu,\kappa)\in\mathsf{Hei}\;:\;\lambda,\mu,\kappa\in\mathbb{Z}\}\,, \tag{8.12}$$

and the inverse image of the modular group $\tilde\Gamma=\mathrm{pr}^{-1}\Gamma$, as defined in §2.3.

Suppose that a function f on $\tilde G^J$ has index m and weight k as indicated in (8.5). If f is left-invariant under Λ, then for $\mathbf{g}\in\tilde G^J$:

$$f(\mathbf{g}) \;=\; f(\mathrm{h}(0,0,1)\mathbf{g}) \;=\; f(\mathbf{g}\mathrm{h}(0,0,1)) \;=\; e^{2\pi im}f(\mathbf{g})\,.$$

So we need m to be integral for invariance under Λ.

The element $\tilde S\;=\;\widetilde{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}=\tilde{\mathrm{k}}(-\pi/2)$ satisfies $\tilde S^4\;=\;\tilde{\mathrm{k}}(-2\pi)\neq 1$, although $\begin{pmatrix}0&-1\\1&0\end{pmatrix}^4$ is the unit matrix. We have

$$f(\tilde S^4\mathbf{g}) \;=\; f(\mathbf{g}\tilde S^4) \;=\; e^{-2\pi ik}f(\mathbf{g})\,.$$

Hence, if $k\in\mathbb{R}\smallsetminus\mathbb{Z}$, then a function cannot be left-invariant under $\tilde\Gamma^J$.

We restrict our attention to left-$\tilde\Gamma^J$-equivariant functions f on $\tilde G^J$ of index $m \in \mathbb{Z}_{\geq 1}$ and weight $k \in \mathbb{R}$ satisfying

$$f(\boldsymbol{\gamma}\mathbf{g}) = \varphi(\boldsymbol{\gamma})\, f(\mathbf{g}) \qquad (\boldsymbol{\gamma} \in \tilde\Gamma^J, \mathbf{g} \in \tilde G^J) \tag{8.13}$$

for a character $\varphi\colon \tilde\Gamma^J \to \mathbb{C}^*$ that satisfies

$$\varphi(\lambda) = 1 \text{ for } \lambda \in \Lambda, \qquad \varphi(\tilde S^4) = e^{-2\pi i k}. \tag{8.14}$$

One such character is χ_k as defined in (2.26) and extended to $\tilde\Gamma^J$ by taking $\chi_k(\lambda) = 1$ for $\lambda \in \Lambda$. All other such characters are of the form $\varphi = \varphi_a \chi_k$ with $a \in \mathbb{Z} \bmod 12$

$$\varphi_a(\tilde T) = e^{\pi i a/6}, \qquad \varphi_a(\tilde S) = e^{-\pi i a/2}. \tag{8.15}$$

The φ_a are trivial on $\tilde Z_2 = \ker \mathrm{pr}$; see (2.16). Hence the φ_a correspond to characters of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

For functions F on $\mathfrak{H} \times \mathbb{C}$ we define the action $|^J_{\varphi_a v_k,k,m}$ of Γ^J on functions on $\mathfrak{H} \times \mathbb{C}$ by

$$\begin{aligned} F|^J_{\varphi_a v_k,k,m}\gamma &= \varphi_a(\gamma)^{-1} v_k(\gamma)^{-1}\, F|^J_{k,m}\gamma && \text{for } \gamma \in \Gamma,\\ F|^J_{\varphi_a v_k,k,m}\lambda &= F|^J_{k,m}\lambda && \text{for } \lambda \in \Lambda. \end{aligned} \tag{8.16}$$

If F corresponds via $\tilde\Psi_{k,m}$ (see (8.9) and (8.10)) to a function f satisfying (8.13), then F is invariant under the action $|_{\varphi_a v_k,k,m}$ of Γ^J. To see this we use the relation $v_k(\gamma) = \chi_k(\ell(\gamma))$ and the fact that φ_a is a character of Γ.

8.3. **Lie algebra.** The group G^J and its covering group $\tilde G^J$ have the same Lie algebra $\mathfrak{g}^J$. We use the notation of basis elements of $\mathfrak{g}^J$ as indicated in [2, §1.3, §1.4]. That are Z, X_+ and X_- in the Lie algebra of G, already used in (2.31), and Z_0, Y_+ and Y_- in the Lie algebra of Hei. Each element of the Lie algebra acts on the functions in $C^\infty(\tilde G^J)$ by right differentiation. For any function f of weight k and index m we have

$$Z_0 f = 2\pi m\, f, \qquad Z f = k\, f. \tag{8.17}$$

Under the relation (8.9), the differential operator on $\tilde G^J$ given by any $\mathbf{X} \in \mathfrak{g}^J$ commutes with left translations. For given index and weight it corresponds to a differential operator on $\mathfrak{H} \times \mathbb{C}$ by the relation in (8.10). We use Pitale's notation $\mathbf{X}^{k,m}$ for this operator. He gives it explicitly in terms of the coordinates $\tau \in \mathfrak{H}$ and $z \in \mathbb{C}$; see [27, p 91, 92]. We see for instance that the kernel of $Y_-^{k,m}$ consists of the functions F on $\mathfrak{H} \times \mathbb{C}$ that are holomorphic in z.

The elements X_+ and X_- in $\mathfrak{g} \subset \mathfrak{g}^J$ shift the weight of functions on $\tilde G$ by ± 2, respectively. To get weight shifting operators on $\tilde G^J$ from $X_\pm$ we need to calibrate them by adding a correction term based on the elements $Y_\pm$ in the Lie algebra of Hei, setting

$$D_\pm = X_\pm \pm \frac{1}{4\pi m} Y_\pm^2. \tag{8.18}$$

These elements are not in the Lie algebra $\mathfrak{g}^J$ itself, but non-commutative polynomials in Lie algebra elements. Pitale gives the corresponding weight shifting differential operators $X_\pm^{k,m}$ on $\mathfrak{H} \times \mathbb{C}$.

More complicated is the Casimir operator $C^{k,m}$, given in [27, (8)]. It corresponds to a non-commutative polynomial of degree 3 in elements of the Lie algebra $\mathfrak{g}^J$. It has the advantage to commute not only with $|_{k,m}^J g$ for all $g \in G^J$, but also with $\mathbf{X}^{k,m}$ for all elements $\mathbf{X}$ of the Lie algebra. See also [2, Proposition 3.1.10]. Instead of requiring functions to be eigenfunctions of $C^{k,m}$ we can require that functions are eigenfunctions of $D_-^{k+2,m} D_+^{k,m}$ with prescribed eigenvalue depending on the weight. This is similar to the relation (2.32) for $\mathrm{SL}_2(\mathbb{R})$, which implies for functions of a given weight that eigenfunctions of Δ are also eigenfunctions of $X_- X_+$.

8.4. **Jacobi Maass forms.** Jacobi Maass forms can be defined as functions on $\mathfrak{H} \times \mathbb{C}$, as Pitale does. One may also view Jacobi Maass forms as function on the Jacobi group, or on the universal covering group of the Jacobi group. Both points of view are connected by the map $\Psi_{k,m}$ in (8.8). We formulate the definition in both ways.

First we work on $\tilde{G}$:

Definition 8.1. Let $k \in \mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{C}$, and $a \in \mathbb{Z}/12\mathbb{Z}$. The space $\mathcal{A}_{k,m}^J(s, \varphi_a \chi_k)$ of *Jacobi Maass forms* on $\tilde{G}^J$ consists of the functions $f \in C^\infty(\tilde{G})$ that satisfy

(a) $f(\mathrm{h}(0,0,r)\mathbf{g}\tilde{\mathrm{k}}(\vartheta)) = e^{2\pi i m r + ik\vartheta} f(\mathbf{g})$ for $\mathbf{g} \in \tilde{G}^J$, $r, \vartheta \in \mathbb{R}$.
(b) $f(\tilde{\gamma}\mathbf{g}) = \varphi_a(\tilde{\gamma})\chi_k(\tilde{\gamma}) f(\mathbf{g})$ for $\tilde{\gamma} \in \tilde{\Gamma}^J$, $\mathbf{g} \in \tilde{G}^J$.
(c) f satisfies the following relations:

$$D_- D_+ f = \frac{4s^2 - (2k+1)^2}{16} f,$$
$$D_+ D_- f = \frac{4s^2 - (2k-3)^2}{16} f, \quad \text{and} \quad Y_- f = 0.$$

(d) $f(\tilde{\mathrm{a}}(t)\mathbf{g}) = \mathrm{O}(t^\alpha)$ as $t \uparrow \infty$, for some $\alpha \in \mathbb{R}_{>0}$, uniform for $\mathbf{g}$ in compact subsets of $\tilde{G}$. (We recall that $\tilde{\mathrm{a}}(t) = \tilde{\mathrm{p}}(it)$, see (2.15).)

The subspace $\mathcal{A}_{k,m}^{J,0}(s, \varphi_a \chi_k)$ of *Jacobi Maass cusp forms* is determined by replacing (d) by the stronger condition

(d′) $f(\tilde{\mathrm{a}}(t)\mathbf{g}) = \mathrm{O}(t^{-\alpha})$ as $t \uparrow \infty$, for all $\alpha \in \mathbb{R}_{>0}$, uniform for $\mathbf{g}$ in compact subsets of $\tilde{G}^J$.

With relation (8.10) we obtain the following reformulation:

Definition 8.2. Let $k \in \mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{C}$, and $a \in \mathbb{Z} \bmod 12$. The space $\mathcal{A}_{k,m}^J(s, \varphi_a v_k)$ of *Jacobi Maass forms* on $\mathfrak{H} \times \mathbb{C}$ consists of the functions $F \in C^\infty(\mathfrak{H} \times \mathbb{C})$ that satisfy

(B) $F|_{\varphi_a v_k, k, m}^J \gamma = F$ for all $\gamma \in \Gamma^J$.

(C) F satisfies the following relations:

$$\begin{aligned} D_-^{k+2,m} D_+^{k,m} F &= \frac{4s^2 - (2k+1)^2}{16} F\,, \\ D_+^{k-2,m} D_-^{k,m} F &= \frac{4s^2 - (2k-3)^2}{16} F\,, \quad \text{and } Y_-^{k,m} F = 0\,. \end{aligned}$$

(D) $F(\mathrm{a}(y) \cdot (\tau, z)) = \mathrm{O}(y^{\alpha})$ as $y \uparrow \infty$ for some $\alpha \in \mathbb{R}_{>0}$, uniform for (τ, z) in compact sets of $\mathfrak{H} \times \mathbb{C}$. (Here $\mathrm{a}(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \in G$.)

The subspace $\mathcal{A}_{k,m}^{J,0}(s, \varphi_a v_k)$ of *Jacobi Maass cusp forms* is determined by replacing (D) by the stronger condition

(D′) $F(\mathrm{a}(y) \cdot (\tau, z)) = \mathrm{O}(y^{-\alpha})$ as $y \uparrow \infty$ for all $\alpha \in \mathbb{R}_{>0}$, uniform for (τ, z) in compact sets of $\mathfrak{H} \times \mathbb{C}$.

We note that there is no part (A) in Definition 8.2 corresponding to part (a) in Definition 8.1. The weight and the index are properties of functions on $\tilde{G}^J$, and have to be fixed in the definition for $\tilde{G}^J$. On the other hand, the weight and the index are not properties of functions on $\mathfrak{H} \times \mathbb{C}$, but parameters in the transformation behavior. We also note that the character χ_k of $\tilde{\Gamma}$ in Definition 8.1 is replaced in Definition 8.2 by the multiplier system v_k on G given by $v_k(g) = \chi_k(\ell(g))$.

There are a number of differences in comparison with Pitale's Definition 3.2 in [27]:

(1) We allow the weight k to be real, instead of only integral.
(2) Pitale seems to allow the eigenvalue λ of $C^{k,m}$ to depend on F. In that way, $J_{k,m}^{nh}$ is not a linear space.
(3) Our spaces of Jacobi Maass forms are in $\hat{J}_{k,m}^{nh}$ in [27, (30)].
    Even if we fix the eigenvalue λ, the space of Jacobi Maass forms has infinite dimension. Pitale does not impose the condition $Y_-^{k,m} F = 0$ in the definition, but imposes it later on.
(4) The characterization in (c) says that Jacobi Maass forms transform under the Lie algebra action in the same way as a vector $w_0 \otimes v_k$ in the principal series representation described in [2, Proposition 3.1.6]. (There the weights are integral, but the formulas describe a Lie algebra module if we let the weight k run through a class in $\mathbb{R}$ mod $2\mathbb{Z}$.)
(5) We can show that $D_+ D_- f = D_- D_+ f + (k - \frac{1}{2}) f$ on functions of weight k and index m that satisfy $Y_- f = 0$. So we can omit the condition on $D_+ D_- f$ in c) and C).
(6) We use the condition of quick decay to characterize Jacobi Maass cusp forms. It takes a consideration of Fourier expansions to get the formulation used by Pitale.

8.5. **Theta decomposition.** According to [27, Theorem 4.6], each $F \in \mathcal{A}^J_{k,m}(s, v_0)$, with $k \in \mathbb{Z}$ and the trivial multiplier system v_0, is of the form

$$
\begin{aligned}
F(\tau, z) &= \sum_{j \bmod 2m} \Theta_{m,j}(\tau, z) F_j(\tau)\,, \\
\Theta_{m,j}(\tau, z) &= (\operatorname{Im}\tau)^{1/4} \sum_{\alpha \equiv j/2m \bmod 1} e^{2\pi i m\tau\alpha^2}\, e^{4\pi i m z\alpha} \qquad (8.19)\\
&= (\operatorname{Im}\tau)^{1/4} \sum_{r \equiv j \bmod 2m} e^{\pi i \tau r^2/2m}\, e^{2\pi i r z}
\end{aligned}
$$

with a vector $(F_j)_{j \bmod 2m}$ which is a vector-valued Maass form of weight $k-\frac{1}{2}$. This opens the way to attach period functions to Jacobi Maass cusp forms by application of Theorem 7.1. In this subsection we check that the decomposition goes through in the case of real weight.

Theta functions on Hei. Let $m \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}/2m$. For each Schwartz function φ on $\mathbb{R}$ the theta function

$$
\vartheta^{\mathsf{Hei}}_{m,j}(\varphi; \mathrm{h}(p, q, r)) = \sum_{\alpha \equiv j/2m\,(1)} e^{2\pi i m(r+q(2\alpha+p))} \varphi(p+\alpha) \qquad (8.20)
$$

is in $C^\infty(\Lambda\backslash\mathsf{Hei})$, and the subspace of functions in $C^\infty(\Lambda\backslash\mathsf{Hei})$ that transform according to the character $\mathrm{h}(0,0,r) \mapsto e^{2\pi i m r}$ consists of the finite sum of the form $\sum_{j \bmod 2m} \vartheta^{\mathsf{Hei}}_{m,j}(\varphi_j)$ with Schwartz functions φ_j.

Actually, the map $(\varphi_j) \mapsto \sum_{j \bmod 2m} \vartheta^{\mathsf{Hei}}_{m,j}(\varphi_j)$ induces a unitary isomorphism $L^2(\mathbb{R})^{2m} \to L^2(\Lambda\backslash\mathsf{Hei})_m$, where the subscript m indicates the subspace given by the character $\mathrm{h}(0,0,r) \mapsto e^{2\pi i m r}$.

Theta functions on $\tilde{G}^J$. Let $f \in C^\infty(\Lambda\backslash\tilde{G}^J)$. Then there are Schwartz function $\xi \mapsto \varphi_j(\tilde{g}, \xi)$ parametrized by $\tilde{g} \in \tilde{G}$ such that

$$
f(h\tilde{g}) = \sum_{j \bmod 2m} \vartheta^{\mathsf{Hei}}_{m,j}(\varphi_j(\tilde{g}, \cdot); h) \qquad h \in \mathsf{Hei},\ \tilde{g} \in \tilde{G}\,. \qquad (8.21)
$$

Let us define the following family φ^J of Schwartz functions on $\mathbb{R}$ parametrized by $\tilde{G}$:

$$
\varphi^J(\tilde{\mathrm{p}}(\tau)\tilde{\mathrm{k}}(\vartheta), \xi) = \operatorname{Im}(\tau)^{1/4} e^{i\vartheta/2}\, e^{2\pi i m\tau\xi^2}\,, \quad \xi \in \mathbb{R}\,. \qquad (8.22)
$$

For each $m \in \mathbb{Z}_{\neq 0}$ and $j \in \mathbb{Z}$ the function in $C^\infty(\tilde{G}^j)$ defined by

$$
\vartheta_{m,j}(h\tilde{g}) = \vartheta^{\mathsf{Hei}}_{m,j}(\varphi^J(\tilde{g}, \cdot); h)\,, \qquad h \in \mathsf{Hei},\ \tilde{g} \in \tilde{G} \qquad (8.23)
$$

is left-invariant under the elements of Λ, has index m and weight $\frac{1}{2}$. With (8.10) the function $\vartheta_{m,j}$ on $\tilde{G}^J$ corresponds to the function on $\mathfrak{H} \times \mathbb{C}$ that is used in (8.19):

$$
\begin{aligned}
&(\tau, z) \mapsto e^{-2\pi i m z \operatorname{Im}(z)/\operatorname{Im}(\tau)}\, \vartheta_{m,j}\Big(\mathrm{h}\Big(\frac{\operatorname{Im} z}{\operatorname{Im}\tau}, z - \frac{\tau \operatorname{Im} z}{\operatorname{Im}\tau}, 0\Big)\tilde{\mathrm{p}}(\tau)\Big) \\
&\quad = \sum_{\alpha \equiv j/2m(1)} e^{4\pi i m\alpha z}\, e^{-4\pi i m\alpha\tau \operatorname{Im}(z)/\operatorname{Im}(\tau)}\, e^{-2\pi i m\tau \operatorname{Im}(z)^2/\operatorname{Im}(\tau)^2} \\
&\qquad \cdot \varphi^J(\tilde{\mathrm{p}}(\tau), \alpha + \operatorname{Im}(z)/\operatorname{Im}(\tau))
\end{aligned}
$$

$$= \operatorname{Im}(\tau)^{1/4} \sum_{\alpha\equiv j/2m(1)} e^{4\pi i m\alpha z}\, e^{-4\pi i m\alpha\tau \operatorname{Im}(z)/\operatorname{Im}(\tau)}\, e^{-2\pi i m\tau \operatorname{Im}(z)^2/\operatorname{Im}(\tau)^2} \cdot e^{2\pi i m\tau(\alpha+\operatorname{Im}(z)/\operatorname{Im}(\tau))^2}$$

$$= \operatorname{Im}(\tau)^{1/4} \sum_{\alpha\equiv j/2m(1)} e^{4\pi i m\alpha z}\, e^{2\pi i m\alpha^2\tau} = \Theta_{m,j}(\tau,z)\,.$$

We start with the generalization of the theta decomposition (8.19) to real weights, working on the group $\tilde{G}^J$ and on the space $\mathfrak{H}\times\mathbb{C}$.

We can check that $\vartheta_{m,j}$ satisfies the conditions (a), (c) and (d) in Definition 8.1, with spectral parameter $s = 1$ or -1 and weight $\frac{1}{2}$. In particular, it satisfies

$$Y_- \vartheta_{m,j} = D_+ \vartheta_{m,j} = D_- \vartheta_{m,j} = 0\,. \tag{8.24}$$

The behavior of $\Theta_{m,j}$ under left-translation is clear for elements of $\Lambda \subset \mathsf{Hei}$. It suffices to consider the generators. From [11, p. 58, 59] we get for the corresponding function on $\mathfrak{H}\times\mathbb{C}$:

$$\Theta_{m,j}(\tau+1,z) = e^{\pi i j^2/2m}\, \Theta_{m,j}(\tau,z)\,, \tag{8.25}$$

$$\begin{aligned}\Theta_{m,j}(-1/\tau, z/\tau) &= \frac{\operatorname{Im}(\tau)^{1/4}}{|\tau|^{1/2}} \sqrt{\frac{\tau}{2mi}} e^{2\pi i m z^2/\tau} \sum_{j' \bmod 2m} e^{-2\pi i j j'/2m} \frac{\Theta_{m,j'}(\tau,z)}{\operatorname{Im}(\tau)^{1/4}}\\ &= (2m)^{-1/2}\, e^{-\pi i/4}\, e^{i\arg(\tau)/2}\, e^{2\pi i m z^2/\tau} \cdot \sum_{j' \bmod 2m} e^{-\pi i j j'/m}\, \Theta_{m,j'}(\tau,z)\,.\end{aligned} \tag{8.26}$$

So we have $\Theta_{m,j}|^J_{1/2,m}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) = e^{\pi i j^2/2m}\Theta_{m,j}$ and

$$\Theta_{m,j}|^J_{1/2,m}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = e^{-\pi i/4} \sum_{j'} \frac{e^{-\pi i j j'/m}}{\sqrt{2m}}\, \Theta_{m,j'}\,.$$

For the functions $\vartheta_{m,j}$ on $\tilde{G}^J$ this implies the transformation behavior under left translation by elements of $\tilde{\Gamma}^J$. With the row vector

$$\vec{\vartheta}_m = (\vartheta_{m,1},\ldots,\vartheta_{m,2m}) \tag{8.27}$$

the transformation behavior is determined by

$$\begin{aligned} L(h)\vec{\vartheta}_m &= \vec{\vartheta}_m && \text{for } h\in\Lambda\,,\\ L(\tilde{T})\vec{\vartheta}_m &= \vec{\vartheta}_m\, M(\tilde{T})\,,\\ L(\tilde{S})\vec{\vartheta}_m &= e^{-\pi i/4}\vec{\vartheta}_m\, M(\tilde{S})\,,\end{aligned} \tag{8.28}$$

where $M(\tilde{T})$ denotes the diagonal matrix with $e^{\pi i j^2/2m}$ at position (j,j), and $M(\tilde{S})$ the symmetric matrix with $(2m)^{-1/2}\, e^{-\pi i j j'/m}$ at position (j,j').

We turn to an arbitrary function $f \in C^\infty(\tilde{G}^J)$ with weight $k\in\mathbb{R}$ and index $m\in\mathbb{Z}_{\ge 1}$. We can write f uniquely in the form

$$f(h\tilde{g}) = \vec{\vartheta}_m(h\tilde{g})\, \vec{h}_m(\tilde{g}) \qquad (h\in\mathsf{Hei},\ \tilde{g}\in\tilde{G})\,, \tag{8.29}$$

with a column vector $\vec{h}_m(\tilde{g}) = (h_1(\tilde{g}), \ldots, h_{2m}(\tilde{g}))$ with $h_j \in C^\infty(\tilde{G})$ of weight $k - 1/2$. We can view $\vec{h}$ as a function on $\tilde{G}^J$ depending only on the second factor in $\mathbf{g} = h\tilde{g} \in \mathsf{Hei} \rtimes \tilde{G}$.

The transformation behavior $L(\tilde{\gamma})f = \varphi_a(\tilde{\gamma})\chi_k(\tilde{\gamma})f$ in Definition 8.1 takes the form

$$\vec{\vartheta}_m(\tilde{\gamma}\mathbf{g})\vec{h}_m(\tilde{\gamma}\mathbf{g}) = \varphi_a(\tilde{\gamma})\chi_k(\tilde{\gamma})\,\vec{\vartheta}_m(\mathbf{g})\vec{h}_m(\mathbf{g}) \qquad (\tilde{\gamma} \in \tilde{\Gamma})\,. \tag{8.30}$$

Since the h_j depend only on the factor $\tilde{g} \in \tilde{G}$ in $\mathbf{g} = h\tilde{g} \in \mathsf{Hei}\,\tilde{G}$, we do not get any condition on $\vec{h}_m$ for $\gamma \in \Lambda$. For $\tilde{T}$ and $\tilde{S}$ we obtain

$$M(\tilde{T})\vec{h}_m(\tilde{T}g) = e^{\pi i(a+k)/6}\,\vec{h}_m(g)\,, \qquad e^{-\pi i/4}\,M(\tilde{S})\vec{h}_m(\tilde{S}g) = e^{-\pi i(a+k)/2}\,\vec{h}_m(g)\,.$$

This implies that $\vec{h}$ has to satisfy the transformation behavior

$$\vec{h}(\tilde{\gamma}\tilde{g}) = \chi_{k-1/2}(\tilde{\gamma})\rho_{a,m}(\tilde{\gamma})\,\vec{h}_m(\tilde{g}) \qquad (\tilde{\gamma} \in \tilde{\Gamma},\ \tilde{g} \in \tilde{G})\,, \tag{8.31}$$

with the representation $\rho_{a,m}$ of $\tilde{\Gamma}$ such that $\rho_{a,m}(\tilde{T})$ is $e^{\pi i(a/6+1/12)}$ times the diagonal matrix with entry $e^{-\pi i j^2/2m}$ at position (j, j) (with $1 \leq j \leq 2m$), and $\rho_{a,m}(\tilde{S})$ is $e^{-\pi i a/2}$ times the symmetric matrix with $e^{\pi i j j'/m}/\sqrt{2m}$ at position (j, j').

We turn to the differential relations in condition (c) in Definition 8.1. The differentiation by Y_- only involves the factor Hei of $\tilde{G}^J$, and sends the components h_j of $\vec{h}_m$ to zero. In view of remark (5) after Definition 8.1 we have to look only at the condition

$$D_-D_+f = \frac{(s-k-1/2)\,(s+k+1/2)}{4}f\,. \tag{8.32}$$

We have

$$\begin{aligned} D_+(\vartheta_{m,j}h_j) &= (X_+ + (4\pi m)^{-1}Y_+^2)(\vartheta_{m,j}h_j) \\ &= (X_+\vartheta_{m,j})\,h_j + \vartheta_{m,j}(X_+h_j) + \frac{1}{4\pi m}\Big((Y_+^2\vartheta_{m,j})\,h_j + 2(Y_+\vartheta_{m,j})(Y_+h_j) + \vartheta_{m,j}Y_+^2h_j\Big) \\ &= (dD_+\vartheta_{m,j})\,h_j + \vartheta_{m,j}(X_+h_j) + 0 \\ &= \vartheta_{m,j}\,(X_+h_j)\,, \end{aligned}$$

where we have used that $Y_+h_j = 0$ and, by (8.24), $D_+\vartheta_{m,j} = 0$. Proceeding in a similar way, we obtain

$$\begin{aligned} D_-D_+(\vartheta_{m,j}h_j) &= (D_-\vartheta_{m,j})\,(X_+h_j) + \vartheta_{m,j}\,(X_-X_+h_j) \\ &= \vartheta_{m,j}\,(X_-X_+h_j)\,. \end{aligned}$$

This means that condition (8.32) is equivalent to the condition

$$X_-X_+h_j = \frac{s-(k-1/2)-1}{2}\,\frac{s+(k-1/2)+1}{2}\,h_j\,. \tag{8.33}$$

With $s_1 = \frac{s+1}{2}$ this becomes and $k_1 = k - \frac{1}{2}$ this becomes

$$X_-X_+h_j = \Big(s_1 + \frac{k_1}{2}\Big)\Big(s_1 - 1 - \frac{k_1}{2}\Big)\,h_j\,. \tag{8.34}$$

In view of (2.32) this is just the differential relation that Maass forms on $\tilde{G}$ of weight $k_1 = k - \frac{1}{2}$ have to satisfy.

Theorem 8.3. *Let $m \in \mathbb{Z}_{\geq 1}$, $k \in \mathbb{R}$, $a \in \mathbb{Z}/12$, $s \in \mathbb{C}$, $\operatorname{Re} s \geq 0$, and put $s' = \frac{s+1}{2}$, $k' = k - \frac{1}{2}$. There is a bijective linear map*

$$V_{m,k,s} : \mathcal{A}^J_{k,m}(s, \varphi_a \chi_k) \to \mathcal{A}_{k'}(s', \rho_{a,m} v_{k'})\,, \tag{8.35}$$

where $\rho_{a,m}$ is the $2m$-dimensional unitary representation of $\Gamma^J = \tilde{\Gamma}^J / \tilde{Z}_2$ determined by

$$\begin{aligned} \rho_{a,m}(h) &= I_{2m} \qquad \text{for } h \in \Lambda\,, \\ \rho_{a,m}(\tilde{T})_{j,j'} &= \delta_{j,j'}\, e^{\pi i(a/6+1/12-j^2/2m)}\,, \\ \rho_{a,m}(\tilde{S})_{j,j'} &= (2m)^{-1/2}\, e^{-\pi i(a/2+jj'/m)}\,, \end{aligned} \tag{8.36}$$

for j and j' running from 1 to $2m$. Furthermore,

$$V_{m,k,s}\, \mathcal{A}^{J,0}_{k,m}(s, \varphi_a \chi_k) = \mathcal{A}^0_{k'}(s', \rho_{a,m} v_{k'})\,.$$

Proof. We have seen already how the equivariance of $f \in \mathcal{A}^J_{k,m}(s, \varphi_a \chi_k)$ is equivalent to the transformation behavior of the vector $\vec{h}$ of functions on $\tilde{G}$ in (8.29); and also that the relations (c) in Definition 8.1 are equivalent to the differential equations in Definition 2.1. What remains to be done is the relation between the growth conditions in both definitions.

The theta series

$$\begin{aligned} &\vartheta_{m,j}(\mathrm{h}(p,q,r)\tilde{\mathrm{p}}(\tau)\tilde{\mathrm{k}}(\vartheta)) \\ &\quad = \operatorname{Im}(\tau)^{1/4} e^{i\vartheta/2} \sum_{\alpha \equiv j/2m \bmod 1} e^{2\pi i m\left(r+q(2\alpha+p)\right)}\, e^{2\pi i m \tau (p+\alpha)^2} \end{aligned} \tag{8.37}$$

has polynomial growth. Hence polynomial growth of all h_j implies polynomial growth of f, and quick decay of all h_j implies quick decay of f.

Consider a fixed value of j. With $p = -\frac{j}{2m}$ the theta series has one term that is a non-zero multiple of $\operatorname{Im} \tau^{1/4}$. Hence polynomial growth of f implies that this h_j has at most polynomial growth. If f is a Jacobi Maass cusp form, then this h_j has quick decay. Doing this for all $2m$ values of j, we get the desired equivalence. □

In combination with Theorem 7.1 we obtain:

Corollary 8.4. *Let $m \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{C}$, $k \in \mathbb{R}$, such that $0 \leq \operatorname{Re} s < 1$ and $s \not\equiv \pm k \bmod 2$, and put $s' = \frac{s+1}{2}$, $k' = k - \frac{1}{2}$. There is a bijective linear map*

$$\mathcal{A}^{J,0}_{k,m}(s, \varphi_a \chi_k) \to \mathsf{FE}^{\omega}_{\rho_{a,m} v_{k'}, s', k'}\,.$$

Finally, we give an explicit formulation of the period map in terms of Jacobi Maass forms as functions on $\mathfrak{H} \times \mathbb{C}$.

Proposition 8.5. *Let $F \in \mathcal{A}^{J,0}_{k,m}(s, \varphi_a v_k)$ as in Definition 8.2, and put*

$$C_j(\tau) := \operatorname{Im}(\tau)^{-1/4} \int_{z=0}^{1} e^{-2\pi i j z} F(\tau, z)\, dz \qquad (j \in \mathbb{Z},\ \tau \in \mathfrak{H})\,. \tag{8.38}$$

Then $F_j(\tau) \coloneqq e^{-\pi i j^2 \tau/2m}\, C_j(\tau)$ *depends only on the class of* j *in* $\mathbb{Z}/2m\mathbb{Z}$*. It is the* j*-th Maass form in the theta decomposition in* (8.19)*. The period function associated to* F *is given by*

$$\left(\int_{\tau=0}^{i\infty} [F_j(\tau), R_{s',k'}(t,\tau)]_k\right)_{1\le j\le 2m}, \tag{8.39}$$

with $s' = \frac{s+1}{2}$, $k' = k - \frac{1}{2}$.

Proof. The theta decomposition (8.19) can be formulated in terms of the Jacobi Maass form F on $\mathfrak{H}\times\mathbb{C}$ and the components of the associated vector-valued Maass form $(F_j)_{j \bmod 2m}$:

$$F(\tau,z) = \sum_{c=1}^{2m} \Theta_{m,c}(\tau,z) F_c(\tau)\,. \tag{8.40}$$

Expanding the theta functions this becomes

$$\begin{aligned} F(\tau,z) &= \sum_{c=1}^{2m} F_c(\tau) \sum_{\substack{j\in\mathbb{Z}\\ j\equiv c \bmod 2m}} e^{\pi i j^2\tau/2m} \mathrm{Im}\,(\tau)^{1/4} e^{2\pi i j z} \\ &= \sum_{j\in\mathbb{Z}} F_j(\tau)\, \mathrm{Im}\,(\tau)^{1/4}\, e^{\pi i j^2 \tau/2m} e^{2\pi i j z}\,. \end{aligned}$$

Here, for $j\in\mathbb{Z}$, the map F_j refers to the unique map F_c with $c \equiv j \bmod 2m$ with $c\in[1,2m]$. This formula can be viewed as a Fourier expansion in z. The Fourier expansion has only terms that are holomorphic in z. This corresponds to $Y_-^{k,m}F = 0$; see [27, p. 91].

In (8.38) we defined $\mathrm{Im}\,(\tau)^{1/4} C_j(\tau)$ as the Fourier coefficient of order j. So

$$C_j(\tau) = e^{\pi i j^2\tau/2m}\, F_j(\tau)\,.$$

This implies the proposition. □

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LIST OF NOTATIONS

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