ADDITIONAL REMARKS ON AUTOMORPHIC FORMS FOR $\Gamma_0(4)$

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This note is written as a companion to §3 of our paper [BFM]. We put it on our websites, without the intention of formal publication.

In [BFM] we relate numerical and theoretical results on the zeros of the Selberg zeta-function for a special group, namely $\Gamma_0(4)$, and for a special one-parameter group of characters.

The computations of these zeros forms the thesis work of Fraczek, [Fr]. It is based on the Selberg zeta-function as a geometric object, connected to the length spectrum of the closed geodesics in the quotient of the upper half-plane by the discrete group $\Gamma_0(4)$. Fraczek computes the Selberg zeta function by approximating the spectrum of a transfer operator. See Chapter 7 of [Fr], in particular equation (7.21) and Proposition 7.4.9.

Many observations in Fraczek's results ask for a theoretical explanation. In [BFM] we present an explanation of the behavior of the zeros of the Selberg zeta-function as the character approaches the trivial character. The Selberg trace formula relates the geometric structure of $\Gamma_0(4)\backslash\mathbb{H}$, namely the length spectrum of the closed geodesics, to the spectral theory of the Hilbert space $L^2(\Gamma_0(4)\backslash\mathbb{H})$, namely the distribution of the eigenvalues of the Laplace operator. This gives a spectral interpretation of the zeros of the Selberg zeta-function. That interpretation is the basis of our explanations in [BFM].

The three sections of [BFM] have a very different nature. In the first one we discuss observations concerning Fraczek's data and formulate theorems explaining these observations. In the second section we carry out analytic work to prove the theorems. We base this analytic work on a list of facts from the spectral theory of automorphic forms. Not all of these facts can be found directly in the literature in the way we need. In Section 3 of [BFM] we discuss how these facts can be derived, applying published results to the special situation we need. In this last section of [BFM] we had to be concise. Here we give more details and work out some of the computations.

1. The group $\Gamma_0(4)$

By $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \Big/ \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \;:\; t \in \mathbb{R}^* \right\}$ we denote the image in $\mathrm{PGL}_2(\mathbb{R})$ of $\left\{ \begin{pmatrix} ta & tb \\ tc & td \end{pmatrix} \;:\; t \in \mathbb{R}^* \right\}$. The group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\mathrm{I}, -\mathrm{I}\}$ acts on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \;:\; \mathrm{Im}\, z > 0\}$ by fractional linear transformations

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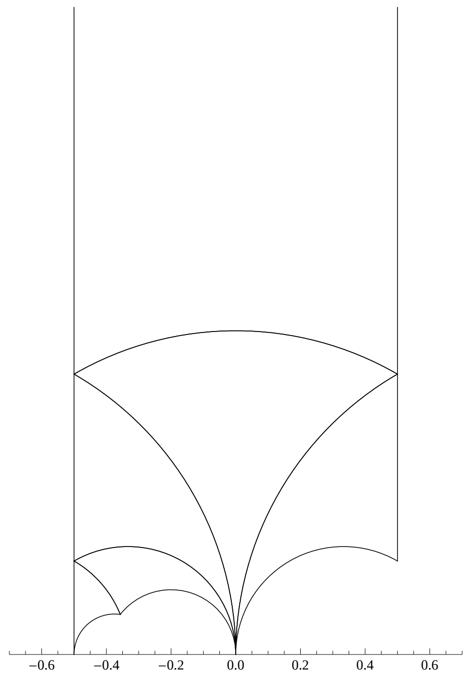

FIGURE 1. Fundamental domain for $\Gamma_0(4)$

$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] z = \frac{az+b}{cz+d}$. We have

$$\Gamma_0(4) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R}) \; ; \; a, b, d \in \mathbb{Z}, \; c \in 4\mathbb{Z} \right\}. \tag{1.1}$$

The group $\Gamma_0(4)$ has index 6 in the modular group $\mathrm{PSL}_2(\mathbb{Z})$. Figure 1 shows a fundamental domain of $\Gamma_0(4)$ consisting of 6 translates of the standard fundamental domain of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

The genus of $\Gamma_0(4)\backslash\mathbb{H}$ is 0, and $\Gamma_0(4)$ has no elliptic elements. The set $\mathbb{Q}\cup\{\infty\}$ of cusps of $\Gamma_0(4)$ consists of three $\Gamma_0(4)$-orbits, for which we use the representatives 0, ∞ and $-\frac{1}{2}$, occurring in the closure of the fundamental domain in the figure. For each cusp $\xi \in \mathbb{P}^1_{\mathbb{Q}}$ we denote by Γ_ξ the subgroup Γ_ξ of $\Gamma_0(4)$ fixing ξ. This group Γ_ξ is generated by $\pi_\xi = g_\xi \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] g_\xi^{-1}$, where $g_\xi \in \mathrm{PSL}_2(\mathbb{R})$ satisfies $\xi = g_\xi \, \infty$. The matrices π_ξ and our choice of g_ξ are as follows:

(1.2)

ξ	0	∞	$-\frac{1}{2}$
π_ξ	$\left[\begin{smallmatrix} 1 & 0 \\ -4 & 1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 3 & 1 \\ -4 & -1 \end{smallmatrix}\right]$
g_ξ	$\left[\begin{smallmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right]$

The elements π_0, π_∞ and $\pi_{-1/2}$ give the boundary identifications in the fundamental domain sketched in Figure 1, hence they generate $\Gamma_0(4)$. They satisfy the relation

$\pi_0\pi_\infty\pi_{-1/2} = 1$. This is the sole relation between these generators, hence the group $\Gamma_0(4)$ is free on π_0 and π_∞. So the character group $\Gamma^\vee$ consists of the characters $\chi_{\alpha,\alpha'}$ with $(\alpha,\alpha') \in \mathbb{C}^2 \bmod \mathbb{Z}^2$ given by

$$\chi_{\alpha,\alpha'}(\pi_\infty) = e^{2\pi i\alpha}, \qquad \chi_{\alpha,\alpha'}(\pi_0) = e^{2\pi i\alpha'}. \tag{1.3}$$

The characters are unitary if and only if $(\alpha,\alpha') \in \mathbb{R}^2 \bmod \mathbb{Z}^2$.

The group $\Gamma_0(4)$ is invariant under conjugation by elements of a subgroup of $\mathrm{PGL}_2(\mathbb{R})$ generated by $\begin{bmatrix} 1 & \frac12 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & -\frac12 \\ 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, inducing symmetries of the characters described by $(\alpha,\alpha') \mapsto (\alpha, -\alpha-\alpha')$, $(\alpha,\alpha') \mapsto (\alpha',\alpha)$, and $(\alpha,\alpha') \mapsto (-\alpha,-\alpha')$. Conjugation with

$$j := \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac12 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac12 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac12 \\ 2 & 0 \end{bmatrix} \tag{1.4}$$

leaves $\chi_{\alpha,0}$ invariant.

In [BFM] we use the one-parameter family $\alpha \mapsto \chi_\alpha := \chi_{\alpha,0}$ of characters. It is special since χ_α is trivial on π_0. For special values of α and α' the character $\chi_{\alpha,\alpha'}$ is "arithmetic", which means that $\chi_{\alpha,\alpha'}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ is determined by congruence conditions on the matrix element a, b, c, d. This is in particular the case for χ_α with $\alpha \in \frac18\mathbb{Z} \bmod \mathbb{Z}$. See [PS]. For general values of α there seems no other way to compute $\chi_\alpha(\gamma)$ than by expressing $\gamma \in \Gamma_0(4)$ in terms of π_0 and π_∞. We do not have a reason to actually carry this out, except in the proof in §3.6 in [BFM].

2. Maass forms

Maass forms of weight zero for $\Gamma := \Gamma_0(4)$ and the unitary character χ_α with $\alpha \in \mathbb{R}$ of $\Gamma_0(4)$ are functions u on $\mathbb{H}$ satisfying

i) *Invariance:* $u(\gamma z) = \chi_\alpha(\gamma)\, u(z)$ for all $\gamma \in \Gamma$.
ii) *Eigenfunction of Laplace operator:* $\Delta u = \lambda\, u$, where $\Delta = -y^2\partial_y^2 - y^2\partial_x^2$ is the Laplace operator for the standard structure of $\mathbb{H}$ as a riemannian space. (We write $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$ throughout this note.) The complex number λ is called the *eigenvalue* of u.
ii) *Polynomial growth:* $u(g_\xi z) = \mathrm{O}(y^a)$ as $y \to \infty$ for all $\xi \in \{0, \infty, -\frac12\}$ for some $a \in \mathbb{R}$.

One needs differentiability in some sense to impose condition ii). Since the differential operator Δ is analytic with analytic coefficients, its eigenfunctions are automatically real-analytic.

It is convenient to parametrize the eigenvalue as $\lambda = \beta(1-\beta)$, with the *spectral parameter* $\beta \in \mathbb{C}$. By $\mathsf{Maass}(\alpha,\beta) = \mathsf{Maass}(\alpha, 1-\beta)$ we denote the linear space of Maass forms for the character χ with eigenvalue $\beta(1-\beta)$.

It is known that $\mathsf{Maass}(\alpha,\beta)$ has finite dimension. See, e.g., Theorem 28, Chap. IV, §2 of [Ma], on p. 190. For most values of β it has a basis consisting of Eisenstein series, corresponding to those cusps $\xi \in \{0, \infty, -\frac12\}$ for which

$\chi_\alpha(\pi_\xi) = 1$. This Eisenstein series is for $\operatorname{Re}\beta > 1$ given by

$$E_\alpha^\xi(z) \;=\; \sum_{\gamma\in\Gamma_\xi\backslash\Gamma} \chi_\alpha(\gamma)^{-1}\,\operatorname{Im}(g_\xi^{-1}\gamma z)^\beta\,, \tag{2.1}$$

and has a meromorphic continuation in $\beta \in \mathbb{C}$.

For some values of β there are also *cusp forms*, for which the condition of polynomial growth is replaced by the more strict condition of *rapid decay:* $u(g_\xi z) = \mathrm{O}(y^a)$ as $y \to \infty$ for all $\xi \in \{0, \infty, -\frac{1}{2}\}$ and *all* $a \in \mathbb{R}$. The values of β for which there are non-zero cusp forms form a discrete subset of the set $(0, 1) \cup (\frac{1}{2} + i\mathbb{R})$. By $\mathsf{Maass}^0(\alpha,\beta)$ we denote the linear space of cusp forms.

The differential operator Δ in condition ii) induces a self-adjoint non-negative operator A_α in the Hilbert space $L^2(\Gamma\backslash\mathbb{H}, \chi_\alpha)$ of classes of functions f on $\mathbb{H}$ satisfying $f(\gamma z) = \chi_\alpha(\gamma) f(z)$ for $\gamma \in \Gamma$ and for which

$$\int_{\Gamma\backslash\mathbb{H}} \left|f(z)\right|^2 \frac{dx\,dy}{y^2} \;<\; \infty\,.$$

The non-zero cusp forms in $\mathsf{Maass}^0(\alpha,\beta)$ are eigenvectors of A_α with eigenvalue $\beta(1-\beta)$. The continuous spectrum of A_α corresponds to a subspace spanned by integrals of the Eisenstein series. Some residues of the Eisenstein series contribute to the discrete spectrum as well.

Subsection 3.3 in [BFM] discusses the significance of the zeros of the Selberg zeta-function for the spectral decomposition of the self-adjoint operator A_α and the singularities of the Eisenstein series.

There arises immediately the problem how the zeros behave when α approaches 0. For the trivial character, $\alpha = 0$, the continuous spectrum has multiplicity 3. As soon as α becomes positive, the multiplicity of the continuous spectrum drops down to 1. So the spectral decomposition of the operator A_0 is drastically different from the spectral decomposition of A_α with $\alpha \in (0, 1)$. It seems hard to explain the results in [Fr] on the basis of the classical spectral theory of automorphic forms.

3. Automorphic forms with a bit of exponential growth

Perturbation theory for this situation needs more tools. In [BFM] we use the analytic perturbation theory as developed in [Br].

The concept of Maass forms is widened, by replacing the condition iii) of polynomial growth by the less restrictive condition of *exponential growth:*

$$u(g_\xi z) \;=\; \mathrm{O}(e^{ay}) \tag{3.1}$$

as $y \to \infty$ for all $\xi \in \{0, \infty, -\frac{1}{2}\}$ for some $a > 0$. Furthermore we do not impose that the character χ_α is unitary; we let α run through a neighborhood of $(-1, 1)$ in $\mathbb{C}$. For non-real α the estimate (3.1) need not be uniform in $x = \operatorname{Re} z \in \mathbb{R}$, but only uniform for x in compact sets. The resulting space of automorphic forms for given α and β has infinite dimension. We can make it finite dimensional by putting a bound on a in the exponential in (3.1).

Actually, in [Br] the growth is not controlled by a bound like that in (3.1), but by putting conditions on the Fourier expansion. Each automorphic form satisfying conditions i) and ii) above has a Fourier expansion at each cusp η:

$$u(g_\eta z) \;=\; \sum_{n\equiv\kappa_\eta(\alpha) \bmod 1} F_n^\eta u(z)\,, \tag{3.2}$$

with $\kappa_0 = 0$, $\kappa_\infty(\alpha) = \alpha$ and $\kappa_{-1/2}(\alpha) = -\alpha$, where $F_n^\eta u$ is a function on $\mathbb{H}$ satisfying $F_n u(z+x') = e^{2\pi i n x'}\, F_n^\eta u(z)$. So $F_n^\eta u(z) = e^{2\pi i n x}\, h(y)$, and h satisfies a second order linear differential equation inherited from property ii). This implies that $F_n^\eta u$ is an element of a two-dimensional linear solution space $W(n,\beta)$ for which we can give several bases in terms of special functions. For most cases such bases are given in §4.2 of [Br]. If $\mathrm{Re}\, n \neq 0$ there is a one-dimensional subspace $W^0(n,\beta)$ of functions with exponential decay as $y \to \infty$. In §4.3 the growth of automorphic forms is controlled by prescribing a finite number of Fourier terms at the cusps 0, ∞ and $-\frac{1}{2}$ that are left free in $W(n,\beta)$, and requiring that all other Fourier terms are in $W^0(n,\beta)$.

In [BFM] we work with α such that $-1 < \mathrm{Re}\,\alpha < 1$. We define the space $A(\alpha,\beta)$ of automorphic forms with eigenvalue $\beta(1-\beta)$ for the character χ_α by the requirement that the Fourier terms $F_n^\eta u$ with $n \neq \kappa_\eta(\alpha)$ are in $W^0(n,\beta)$.

We have $\chi_{\alpha+1} = \chi_\alpha$. However, there is monodromy, and $A(\alpha,\beta) \neq A(\alpha+1,\beta)$ for $-1 < \mathrm{Re}\,\alpha < 0$. This presence of monodromy also forces us to stay in the region $|\mathrm{Re}\,\alpha| < 1$, when using this present growth condition.

4. Eisenstein families

The main purpose of [Br] is the study of meromorphic families of automorphic forms depending on (α,β) with value in $A(\alpha,\beta)$ at the points (α,β) at which the family is holomorphic. In [Br] the scope is wider: families depending on (χ,β), where χ runs through the character group of the discrete subgroup, or even through the group of multiplier systems, in which case the weight of the automorphic forms in the family may vary. This wider scope may complicate the specialization of results in [Br] to the situation considered in [BFM].

The Eisenstein series E_α^∞ and $E_\alpha^{-1/2}$ disappear as soon as $\alpha \in (0,1)$. The advantage of the use of analytic perturbation theory is the fact that all three Eisenstein series can be embedded in meromorphic families depending on α and β jointly. It needs quite some work to do this. The essential tool is the pseudo Laplace operator of Colin de Verdière, [CdV]. This leads to a family of operators to which the analytic perturbation theory in Kato's book, [Ka], can be applied. This family has a compact resolvent, which is meromorphic in the character and the spectral parameter.

The basic result thereby is Theorem 10.2.1 in [Br]. It gives the existence of the families E^0, E^∞ and $E^{-1/2}$ in (3.3) of [BFM]. Since the character group of Γ contains not only the χ_α, we need the restriction discussed in 10.2.2 of [Br]. The space V_r considered there is equal to $\mathbb{R}$ in the present situation, the element φ_0 is 0, and the intersection $J \cap V_r$ is $(-1,1)$. The theorem gives three families E^ξ,

$\xi \in \{0, \infty, -\frac{1}{2}\}$ for which the Fourier terms $F^{\eta}_{\kappa_\eta(\alpha)} E^{\xi}$ of "order 0" have a prescribed expression with respect to the basis vectors $\mu(\kappa_\eta(\alpha), \beta)$ and $\mu(\kappa_\eta(\alpha), 1-\beta)$ in (3.4) of [BFM]. We note that $\mu(0,\beta;z) = y^\beta$, which is known from the Fourier expansion of Eisenstein series. The nice fact is that these families have restrictions $\beta \mapsto E^{\xi}(0,\beta)$ which are the meromorphic continuations of the Eisenstein series in (2.1). It is not obvious that this restriction exist. Meromorphic families have singularities along curves, and the line $\alpha = 0$ might be one of these curves.

The coefficients in the expression for the Fourier coefficients of order zero of the Eisenstein series $E^{\xi}_0(\beta)$ are meromorphic functions of β. These coefficients form the *scattering matrix*; see [LP] for an explanation of this terminology. Theorem 10.2.1 in [Br] enables us to embed the scattering matrix in a family $(\alpha,\beta) \mapsto \mathbf{C}(\alpha,\beta)$ of matrices with similar properties. This makes it possible to follow the zeros of the Selberg zeta-function in the region $\operatorname{Re}\beta < \frac{1}{2}$, $\operatorname{Im}\beta > 0$.

5. Poincaré series

It is known that the spaces of cusp forms $\mathsf{Maass}^0(\alpha,\beta)$ with $\alpha \in \mathbb{R}$ are spanned by residues of Poincaré series. (See Satz 6.9 in [Ne], or §11.3 in [Br].) To get hold on the behavior of zeros of the Selberg zeta-function with $\operatorname{Re}\beta = \frac{1}{2}$, we use the Poincaré series in (3.8) of [BFM]. These Poincaré series also can be embedded in families that are meromorphic in (α,β), with α in a neighborhood of $(0,1)$ in $\mathbb{C}$. For them we use another basis for the Fourier terms of order zero at ∞ and $-\frac{1}{2}$. This basis is given in (3.4) and (3.7) of [BFM]. Another application of Theorem 10.2.1 and the specialization in §10.2.2 of [Br] gives the families P^{ξ} in §3.2 of [BFM].

The uniqueness in Theorem 10.2.1 implies that we can express the Eisenstein families and the Poincaré families P^{ξ} in terms of each other, on the region in $\mathbb{C}^2$ where they are both defined. The approach is given in 10.3.4 of [Br]. In [BFM] we leave the actual computations to the reader. In the next section we sketch the computations for the present situation.

6. Relations

It is convenient to describe the Fourier terms of order zero as a column vector:

$$\mathbf{F}_\alpha f = \begin{pmatrix} F^0_0 f \\ F^\infty_\alpha f \\ F^{-1/2}_{-\alpha} f \end{pmatrix} \qquad \text{for } f \in A(\alpha,\beta)\,. \tag{6.1}$$

6.1. **Eisenstein families.** We combine the three Eisenstein families into a row vector

$$\mathsf{Eis}(\alpha,\beta) = \Big(E^0(\alpha,\beta),\ E^\infty(\alpha,\beta),\ E^{-1/2}(\alpha,\beta)\Big), \tag{6.2}$$

which defines a meromorphic family of vectors in $A(\alpha,\beta)^3$.

To describe the Fourier term of order zero we use the diagonal matrix

$$\mathbf{m}(\alpha,\beta) = \begin{pmatrix} \mu(0,\beta) & 0 & 0 \\ 0 & \mu(\alpha,\beta) & 0 \\ 0 & 0 & \mu(-\alpha,\beta) \end{pmatrix}, \tag{6.3}$$

with the meromorphic family $\mu(\cdot,\cdot)$ of functions as given in (3.4) of [BFM]. Then

$$\mathbf{F}_\alpha \mathsf{Eis}(\alpha,\beta) \;=\; \mathbf{m}(\alpha,\beta) + \mathbf{m}(\alpha,1-\beta)\,\mathbf{C}(\alpha,\beta)\,, \tag{6.4}$$

as an identity between matrices of meromorphic functions. The matrix $\mathbf{C}(\alpha,\beta)$ is the extended scattering matrix in (2.2) of [BFM]. (We note that here and in the sequel we often suppress the variable $z \in \mathbb{H}$. All terms in (6.4) depend on z, except the scattering matrix $\mathbf{C}(\alpha,\beta)$.)

To get the functional equation for $\beta \mapsto 1-\beta$ we use that $A(\alpha,1-\beta) = A(\alpha,\beta)$. So $\mathsf{Eis}(\alpha,1-\beta)$ is a meromorphic family of automorphic forms in $A(\alpha,\beta)^3$ with

$$\mathbf{F}_\alpha \mathsf{Eis}(\alpha,1-\beta) \;=\; \mathbf{m}(\alpha,\beta)\,\mathbf{C}(\alpha,1-\beta) + \mathbf{m}(\alpha,1-\beta)\,.$$

The uniqueness statement in Theorem 10.2.1 of [Br] implies that

$$\mathsf{Eis}(\alpha,1-\beta) \;=\; \mathsf{Eis}(\alpha,\beta)\,\mathbf{C}(\alpha,1-\beta)\,,$$

and hence

$$\mathrm{Id} \;=\; \mathbf{C}(\alpha,\beta)\,\mathbf{C}(\alpha,1-\beta)\,. \tag{6.5}$$

So $\mathbf{C}(\alpha,\beta)$ is invertible as a matrix of meromorphic functions, with $\mathbf{C}(\alpha,1-\beta)$ as its inverse.

We leave to the reader the derivation, in a similar way, of the relation $\overline{\mathbf{C}(\bar\alpha,\bar\beta)} = \mathbf{C}(-\alpha,\beta)$; for this one has to look at the effect of conjugation on $\mu(\alpha,\beta;z)$. The Maass-Selberg relations lead to the functional equation $\mathbf{C}(-\alpha,\beta) = \mathbf{C}(\alpha,\beta)^t$ (transpose), in the same way as in §11.2.1 of [Br].

6.2. **Poincaré families.** We put also the three Poincaré families into a row vector:

$$\mathsf{Poinc}(\alpha,\beta) \;=\; \Big(P^0(\alpha,\beta),\; P^\infty(\alpha,\beta),\; P^{-1/2}(\alpha,\beta)\Big)\,. \tag{6.6}$$

The special form of its Fourier expansion is computed with respect to another basis. For the Fourier term $F^0_{\kappa_0(\alpha)}P^\xi = F^0_0 P^\xi$ we are forced to use $\mu(\alpha,\mu)$ and $\mu(\alpha,1-\mu)$. For $F^\eta_{\kappa_\eta(\alpha)}P^\xi$ with $\eta = \infty$ or $-\frac{1}{2}$ we use the basis $\mu(\kappa_\eta(\alpha),\beta)$, $\omega(\kappa_\eta(\alpha),\beta)$, with ω as given in (3.7) in [BFM]. This is summarized in the matrix

$$\mathbf{w}(\alpha,\beta) \;=\; \begin{pmatrix} \mu(0,1-\beta) & 0 & 0 \\ 0 & \omega(\alpha,\beta) & 0 \\ 0 & 0 & \omega(-\alpha,\beta) \end{pmatrix}. \tag{6.7}$$

Now we have a 3×3-matrix $\mathbf{D}(\alpha,\beta)$ of meromorphic functions such that

$$\mathbf{F}_\alpha \mathsf{Poinc}(\alpha,\beta) \;=\; \mathbf{m}(\alpha,\beta) + \mathbf{w}(\alpha,\beta)\,\mathbf{D}(\alpha,\beta)\,. \tag{6.8}$$

6.3. **Relation.** The uniqueness of the Eisenstein families and of the Poincaré families implies that they can be related on their common domain.

We have to relate $\mathbf{w}(\alpha,\beta)$ and $\mathbf{m}(\alpha,\beta)$. For $\operatorname{Re} n \neq 0$ we have the identity

$$\begin{aligned} \omega(n,\beta) &= \upsilon(n,\beta)\,\mu(n,\beta) + \upsilon(n,1-\beta)\,\mu(n,1-\beta)\,, \\ \upsilon(n,\beta) &= \pi^{-1/2}(\pi n\;\mathrm{Sign}\,n)^\beta\,\Gamma\Big(\frac{1}{2}-\beta\Big)\,, \end{aligned} \tag{6.9}$$

which is just a relation between confluent hypergeometric functions. To put the relation between the bases into matrix notation we put

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{V}(\alpha,\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & v(\alpha,\beta) & 0 \\ 0 & 0 & v(\alpha,\beta) \end{pmatrix}, \tag{6.10}$$

and then arrive at

$$\mathbf{w}(\alpha,\beta) = \mathbf{m}(\alpha,\beta)\,\mathbf{P}\,\mathbf{V}(\alpha,\beta) + \mathbf{m}(\alpha,1-\beta)\,\mathbf{V}(\alpha,1-\beta)\,. \tag{6.11}$$

The uniqueness of $\mathsf{Eis}(\alpha,\beta)$ implies that $\mathsf{Poinc}(\alpha,\beta) = \mathsf{Eis}(\alpha,\beta)\,\mathbf{W}(\alpha,\beta)$ for some meromorphic family of 3×3-matrices on a neighborhood of $(0,1)$ in $\mathbb{C}$ times $\mathbb{C}$. Uniqueness of Poinc implies that $\mathbf{W}(\alpha,\beta)$ is invertible (as a meromorphic family of matrices, where however invertibility may fail on a subset of lower dimension). Hence

$$\begin{aligned} \mathbf{m}(\alpha,\beta) + \mathbf{w}(\alpha,\beta)\,\mathbf{D}(\alpha,\beta) &= \Big(\mathbf{m}(\alpha,\beta) + \mathbf{m}(\alpha,1-\beta)\,\mathbf{C}(\alpha,\beta)\Big)\,\mathbf{W}(\alpha,\beta) \\ &= \mathbf{m}(\alpha,\beta)\Big(\mathbf{I} - \mathbf{P}\,\mathbf{V}(\alpha,\beta)\,\mathbf{V}(\alpha,1-\beta)^{-1}\mathbf{C}(\alpha,\beta)\Big)\,\mathbf{W}(\alpha,\beta) \\ &\quad + \mathbf{w}(\alpha,\beta)\,\mathbf{V}(\alpha,1-\beta)^{-1}\,\mathbf{C}(\alpha,\beta)\,\mathbf{W}(\alpha,\beta)\,. \end{aligned}$$

This implies

$$\begin{aligned} \mathbf{W}(\alpha,\beta) &= \Big(\mathbf{I} - \mathbf{P}\,\mathbf{V}(\alpha,\beta)\,\mathbf{V}(\alpha,1-\beta)^{-1}\,\mathbf{C}(\alpha,\beta)\Big)^{-1}, \\ \mathbf{D}(\alpha,\beta) &= \mathbf{V}(\alpha,1-\beta)^{-1}\,\mathbf{C}(\alpha,\beta)\,\mathbf{W}(\alpha,\beta) \end{aligned} \tag{6.12}$$

6.4. **Parity.** The map $\iota : z \mapsto \bar{z}/(2\bar{z}-1)$ in (3.5) in [BFM] induces an involution J in each of the spaces $A(\alpha,\beta)$. For the Fourier terms of order zero one then finds

$$\mathbf{F}_\alpha J f(z) = \mathbf{J}\mathbf{F}_\alpha f(-\bar{z})\,, \qquad \mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.13}$$

Hence

$$\begin{aligned} \mathbf{F}_\alpha J\mathsf{Eis}(\alpha,\beta)(z) &= \mathbf{J}\,(\mathbf{m}(\alpha,\beta;-\bar{z}) + \mathbf{m}(\alpha,1-\beta;-\bar{z})\,\mathbf{C}(\alpha,\beta)) \\ &= \mathbf{J}\,(\mathbf{m}(-\alpha,\beta;z) + \mathbf{m}(-\alpha,1-\beta;z)\,\mathbf{C}(\alpha,\beta)) \\ &= \mathbf{m}(\alpha,\beta;z)\,\mathbf{J} + \mathbf{m}(\alpha,1-\beta;z)\,\mathbf{J}\,\mathbf{C}(\alpha,\beta)\,. \end{aligned}$$

So by uniqueness

$$J\mathsf{Eis}(\alpha,\beta) = \mathsf{Eis}(\alpha,\beta)\,\mathbf{J}\,, \qquad \mathbf{J}\,\mathbf{C}(\alpha,\beta)\,\mathbf{J} = \mathbf{C}(\alpha,\beta)\,, \tag{6.14}$$

leading to relations among the different matrix elements of the extended scattering matrix $\mathbf{C}(\alpha,\beta)$.

6.5. **Partial diagonalization.** The eigenspaces of J in $A(\alpha,\beta)$ give a direct sum decomposition into spaces of even and odd automorphic forms. With the unitary matrix $\mathbf{U}$ in (2.7) of [BFM], the first and second coordinate of $\mathsf{Eis}(\alpha,\beta)\,\mathbf{U}^{-1}$ and $\mathsf{Poinc}(\alpha,\beta)\,\mathbf{U}^{-1}$ are even and the third one is odd. Similarly, the first and second coordinate of $\mathbf{UF}_\alpha f$ can be non-zero only if the automorphic form f is even; the third coordinate can be non-zero only if f is odd.

In this way we arrive at the partial diagonalization of the extended scattering matrix

$$\begin{aligned}
\text{(6.15)} \qquad \mathbf{U}\,\mathbf{C}(\alpha,\beta)\,\mathbf{U}^{-1} &= \begin{pmatrix} C_{0,0}(\alpha,\beta) & \sqrt{2}\,C_{0,\infty}(\alpha,\beta) & 0 \\ \sqrt{2}\,C_{\infty,0}(\alpha,\beta) & C_+(\alpha,\beta) & 0 \\ 0 & 0 & C_-(\alpha,\beta) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{C}_+(\alpha,\beta) & 0 \\ 0 & C_-(\alpha,\beta) \end{pmatrix}, \\
\text{(6.16)} \qquad C_\pm(\alpha,\beta) &= C_{\infty,\infty}(\alpha,\beta) \pm C_{\infty,-1/2}(\alpha,\beta)\,.
\end{aligned}$$

The 2×2-matrix $\mathbf{C}_+$ and the 1×1-matrix C_- inherit the properties of $\mathbf{C}$ in (2.3) of [BFM].

6.6. **Further computations.**

$$\text{(6.17)} \qquad \begin{aligned}
&\mathbf{U}\,\mathbf{P}\,\mathbf{V}(\alpha,\beta)\,\mathbf{V}(\alpha,1-\beta)^{-1}\,\mathbf{C}(\alpha,\beta)\mathbf{U}^{-1} \\
&= \mathbf{P}\,\mathbf{V}(\alpha,\beta)\,\mathbf{V}(\alpha,1-\beta)^{-1} \begin{pmatrix} \mathbf{C}_+(\alpha,\beta) & 0 \\ 0 & C_-(\alpha,\beta) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}XC_{\infty,0} & XC_+ & 0 \\ 0 & 0 & XC_- \end{pmatrix},
\end{aligned}$$

with

$$\text{(6.18)} \qquad X = X(\alpha,\beta) = v(\alpha,\beta)\,v(\alpha,1-\beta)^{-1}\,,$$

as in (2.4) in [BFM]. Hence

$$\text{(6.19)} \qquad \mathbf{U}\,\mathbf{W}(\alpha,\beta)^{-1}\,\mathbf{U}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2}XC_{\infty,0} & 1-XC_+ & 0 \\ 0 & 0 & 1-XC_- \end{pmatrix},$$

and

$$\text{(6.20)} \qquad \mathbf{U}\,\mathbf{W}(\alpha,\beta)\,\mathbf{U}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\sqrt{2}XC_{\infty,0}}{1-XC_+} & \frac{1}{1-XC_+} & 0 \\ 0 & 0 & \frac{1}{1-XC_-} \end{pmatrix}$$

This leads to

$$\begin{aligned}
&\mathbf{U}\,\mathbf{D}(\alpha,\beta)\,\mathbf{U}^{-1}\\
&= \mathbf{V}(\alpha,1-\beta)^{-1}\begin{pmatrix}\mathbf{C}_+ & 0\\ 0 & C_-\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ \frac{\sqrt{2}XC_{\infty,0}}{1-XC_+} & \frac{1}{1-XC_+} & 0\\ 0 & 0 & \frac{1}{1-XC_-}\end{pmatrix}\\
&= \mathbf{V}(\alpha,1-\beta)^{-1}\begin{pmatrix} C_{0,0}+\frac{2XC_{0,\infty}C_{\infty,0}}{1-XC_+} & \frac{\sqrt{2}C_{0,\infty}}{1-XC_+} & 0\\ \sqrt{2}C_{\infty,0}+\frac{\sqrt{2}XC_{\infty,0}C_+}{1-XC_+} & \frac{C_+}{1-XC_+} & 0\\ 0 & 0 & \frac{C_-}{1-XC_-}\end{pmatrix}.
\end{aligned}\tag{6.21}$$

The element in the upper left corner is

$$\frac{C_{0,0}-X\ \det\mathbf{C}_+}{1-XC_+} = \frac{C_{0,0}-X(C_{0,0}C_+-2C_{0,\infty}^2)}{1-XC_+},$$

where we use the equality $C_{0,\infty}=C_{\infty,0}$, proved in §3.6 in [BFM]. We arrive at

$$\mathbf{U}\,\mathbf{D}(\alpha,\beta)\,\mathbf{U}^{-1} = \begin{pmatrix}\frac{C_{0,0}-X\ \det\mathbf{C}_+(\alpha,\beta)}{1-XC_+} & \frac{\sqrt{2}\,C_{0,\infty}}{1-XC_+} & 0\\ \frac{\sqrt{2}\,C_{\infty,0}}{\tilde{v}(1-XC_+)} & \frac{C_+}{\tilde{v}(1-XC_+)} & 0\\ 0 & 0 & \frac{C_-}{\tilde{v}\,(1-X\,C_-)}\end{pmatrix},\tag{6.22}$$

with $\tilde{v}=v(\alpha,1-\beta)$.

Under the conjugation $\mathbf{D}(\alpha,\beta)\mapsto\mathbf{U}\mathbf{D}(\alpha,\beta)\mathbf{U}^{-1}$ the matrix element in the upper left corner does not change. This gives equation (2.6) in [BFM].

7. Singularities of the extended scattering matrix

Fact **F4** in §2 of [BFM] concerns a singularity of the extended scattering matrix on the central line $\frac{1}{2}+i\mathbb{R}$ for the unperturbed character, $\alpha=0$. It states that such a singularity leads to a zero of the unperturbed Selberg zeta-function $Z(0,\cdot)$, and cannot occur at $(\alpha,\beta)=(0,\frac{1}{2})$.

The proof of this fact, as briefly sketched in §3.5 of [BFM], shows that if $\mathbf{C}$ is singular at $(0,\beta_0)$ with $\beta_0\in\frac{1}{2}+i[0,\infty)$ then the space $\mathsf{Maass}^0(0,\frac{1}{2})$ is non-zero. Now we can use a) in §3.3 of [BFM]. By [Hu] we know that $\mathsf{Maass}^0(0,\frac{1}{2})=\{0\}$, hence $\beta_0\neq\frac{1}{2}$.

First we give the reasoning in the case that $\beta_0\in\frac{1}{2}+i\mathbb{R}$. By the functional equations in §6.1 it suffices to consider $\beta_0\in\frac{1}{2}+i(0,\infty)$. In this case the functions $(\alpha,\beta)\mapsto\mu(\alpha,\beta)$ and $(\alpha,\beta)\mapsto\mu(\alpha,1-\beta)$ form a basis of $W(\alpha,\beta)$ for all (α,β) in a neighborhood of $(0,\beta_0)$ in $\mathbb{C}^2$.

So if the extended scattering matrix is singular at $(0,\beta_0)$ at least one of the components of the vector $\mathsf{Eis}=(E^0,E^\infty,E^{-1/2})$ has a singularity at $(0,\beta_0)$. There are non-zero holomorphic functions ψ on a neighborhood of $(0,\beta_0)$ in $\mathbb{C}^2$ such that $\psi\cdot\mathsf{Eis}$ is holomorphic on a neighborhood of $(0,\beta_0)$. We choose ψ minimal with respect to divisibility in the germ of holomorphic functions at $(0,\beta_0)$ to have $\psi\cdot\mathsf{Eis}$ not identically zero along the zero set of ψ. Since $\beta\mapsto\mathsf{Eis}(0,\beta)$ is holomorphic at β_0 the zero set of ψ intersects $\{0\}\times\mathbb{C}$ discretely. In such a situation §12.1 of [Br] shows that we have a "local curve" through $(0,\beta_0)$ along the zero set of ψ: this is a map $w \mapsto (w^p, \beta_0 + h(w))$, with $p \in \mathbb{N}$ and h holomorphic on a neighborhood of 0 in $\mathbb{C}$, $h(0) = 0$, such that $w \mapsto \psi(w^p, \beta_0 + h(w))$ is identically zero.

Thus, $\mathbf{f}(w) := (\psi \cdot \mathsf{Eis})(w^p, \beta_0 + h(w))$ defines a holomorphic family of vectors of automorphic forms on a neighborhood of 0 in $\mathbb{C}$, with values in $A(w^p, \beta_0 + h(w))^3$. It is not the zero family, by the minimal choice of ψ. It may have a zero at $w = 0$. However, there is $k \in \mathbb{Z}_{\geq 0}$ such that the principal part

$$\mathbf{f}_0 := \lim_{w \neq 0} w^{-k}\, \mathbf{f}(w)$$

is a non-zero element of $A(0, \frac{1}{2})^3 = \mathsf{Maass}(0, \frac{1}{2})^3$. We have

$$\mathbf{F}_{w^p}\mathbf{f}(w) \;=\; 0 \cdot \mathbf{m}(w^p, \beta_0 + h(w)) + \mathbf{m}(w^p, 1 - \beta_0 - h(w))\,(\psi\, \mathbf{C})(w^p, \beta_0 + h(w))\,.$$

So the three Fourier terms of order 0 of any component of $\mathbf{f}_0$ are multiples of $\mu(0, 1-\beta_0; z) = y^{1-\beta_0}$. Hence these components are not in the linear space spanned by the Eisenstein series $E_0^{\xi}(\beta_0)$, since $\mu(0,\beta_0)$ occurs in their Fourier term $F_0^{\xi} E_0^{\xi}(\beta_0)$. The Maass-Selberg relations, as discussed in, e.g., §4.6 of [Br], induce a non-degenerate bilinear form on the space $\mathsf{Maass}(0,\beta_0)/\mathsf{Maass}^0(0,\beta_0)$; the $E_0^{\xi}(\beta_0)$ induce a basis of this space. So $\mathbf{f}_0$ is a cusp form. This is what we want.

In the case $\beta_0 = \frac{1}{2}$ the μ's do not give a basis of $W(0, \frac{1}{2})$. We use the family λ given by

$$\lambda(\alpha,\beta; z) \;=\; \frac{\mu(\alpha,\beta; z) - \mu(\alpha, 1-\beta; z)}{2\beta - 1}$$

for $\beta \neq \frac{1}{2}$. It extends as a holomorphic family on a neighborhood of $(0, \frac{1}{2})$ in $\mathbb{C}^2$, with value $y^{1/2} \log y$ at $(\alpha,\beta) = (0, \frac{1}{2})$. The functions $\lambda(\alpha,\beta)$ and $\mu(\alpha, 1-\beta)$ form a basis of $W(\alpha,\beta)$ for all (α,β) in a neighborhood of $(0, \frac{1}{2})$. See, e.g., Lemma 7.6.14 i) in [Br]. With

$$\mathbf{l}(\alpha,\beta) \;=\; \begin{pmatrix} \lambda(0,\beta) & 0 & 0 \\ 0 & \lambda(\alpha,\beta) & 0 \\ 0 & 0 & \lambda(-\alpha,\beta) \end{pmatrix} \tag{7.1}$$

and $\mathbf{m}(\alpha,\beta)$ as in (6.3) we have

$$\mathbf{F}_\alpha \mathsf{Eis}(\alpha,\beta) \;=\; (2\beta - 1)\, \mathbf{l}(\alpha,\beta) + \mathbf{m}(\alpha, 1-\beta)\, (\mathbf{I} + \mathbf{C}(\alpha,\beta))\,. \tag{7.2}$$

The family of matrices $\beta \mapsto \mathbf{I} + \mathbf{C}(0,\beta)$ is holomorphic at $\beta = \frac{1}{2}$, and is the zero matrix at $\beta = \frac{1}{2}$, as we see from the explicit expression for the scattering matrix in equation (2.1) of [BFM]. So $\mathbf{F}_0\mathsf{Eis}(0,\beta)$ tends to zero as $\beta \to \frac{1}{2}$. Proposition 10.2.14 implies that then also $\mathsf{Eis}(0,\beta)$ tends to the zero vector of automorphic forms. We consider then

$$\mathbf{e}^{\xi} \;=\; \lim_{\beta \to 1/2} \frac{1}{2\beta - 1}\, E_0^{\xi}(\beta) \qquad (\xi \in \{0, \infty, -\tfrac{1}{2}\})\,. \tag{7.3}$$

This gives three Maass forms in $\mathsf{Maass}(0, \frac{1}{2})$ with Fourier terms of order zero of the form

$$F_0^{\eta} \mathbf{e}^{\xi}(z) \;=\; \delta_{\eta,\xi}\, \lambda(0, \tfrac{1}{2}; z) + c_{\eta,\xi}\, \mu(0, \tfrac{1}{2}; z) \;=\; (\delta_{\eta,\xi}\, \log y + c_{\eta,\xi})\, y^{1/2}\,. \tag{7.4}$$

Hence the $\mathbf{e}^{\xi}$ span a three-dimensional subspace of $\mathsf{Maass}(0, \frac{1}{2})$, which is all of $\mathsf{Maass}(0, \frac{1}{2})$ by the Maass-Selberg relations and Huxley's eigenvalue estimate in [Hu], which implies that $\mathsf{Maass}^0(0, \frac{1}{2}) = \{0\}$.

The extended scattering matrix $\mathbf{C}$, and hence $\mathbf{I}+\mathbf{C}$, might still have a singularity at $(0, \frac{1}{2})$. We proceed as in the case $\beta_0 \in \frac{1}{2} + i(0,\infty)$, now however with the basis $\lambda(\alpha,\beta)$, $\mu(\alpha, 1-\beta)$. This leads to a non-zero vector of Maass forms $\mathbf{f}_0 \in \mathsf{Maass}(0, \frac{1}{2})^3$ for which all Fourier terms of order zero are multiples of $\mu(0, \frac{1}{2}; z) = y^{1/2}$. It cannot be expressed in the basis $\{\mathbf{e}^{\xi}\}$, which gives a contradiction to what we arrived at in the previous paragraph.

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