18b. Coefficient relations

In[ ]:=
```
Clear[coI, coK, fr, f]
fr[r_, t_] =
 t^(p + 2) (coI[r] BesselI[(h - r) / 2, 2 Pi Abs[beta] t] + coK[r] BesselK[(h - r) / 2, 2 Pi Abs[beta] t])
```

Out[ ]=
$$t^{2+p}\left(\text{BesselI}\left[\frac{h-r}{2},\ 2\,\pi\,t\,\text{Abs[beta]}\right]\text{coI}[r]+\text{BesselK}\left[\frac{h-r}{2},\ 2\,\pi\,t\,\text{Abs[beta]}\right]\text{coK}[r]\right)$$

Kernel relation for the downward shift operators

In[ ]:=
```
eq1 = (f[r + 2, t] /. r3m1[r + 2]) - f[r + 2, t] // Simplify
eq2 = (f[r, t] /. rm3m1[r]) - f[r, t] // Simplify
```

Out[ ]=
$$-f[2+r,\ t]+\frac{i\left((-4+h-2\,p-r)\,f[r,\ t]+2\,t\,f^{(0,1)}[r,\ t]\right)}{4\,\text{betac}\,\pi\,t}$$

Out[ ]=
$$-f[r,\ t]+\frac{i\left((2+h+2\,p-r)\,f[2+r,\ t]-2\,t\,f^{(0,1)}[2+r,\ t]\right)}{4\,\text{beta}\,\pi\,t}$$

Both these quantities have to be zero.

Substitution of the solutions;

we write $\beta=\rho\, e^{i\,\vartheta}$ and try to get a simple expression.

In[ ]:=
```
Clear[rho, th]
{eq1a, eq2a} = {eq1, eq2} /. f[rr_, t] ⧴ fr[rr, t] /. f^(0,1)[rr_, t] ⧴ D[fr[rr, t], t] //.
   {beta → rho E^(I th), betac → rho E^(-I th), Abs[rho] → rho, Im[th] → 0,
    (1 / pp_)^ee_ ⧴ pp^(-ee), (pp_ rho)^ee_ ⧴ pp^ee rho^ee} // Simplify
```

Out[ ]=

$$
\begin{aligned}
\Big\{ & \frac{1}{4\pi\,\mathrm{rho}} \\
& i\, t^{1+p}\Big(e^{i\,\mathrm{th}}\Big(2\pi\,\mathrm{rho}\,t\,\mathrm{BesselI}\Big[\frac{1}{2}\times(-2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]+(h-r)\,\mathrm{BesselI}\Big[\frac{h-r}{2},\ 2\pi\,\mathrm{rho}\,t\Big]+2 \\
& \quad \pi\,\mathrm{rho}\,t\,\mathrm{BesselI}\Big[\frac{1}{2}\times(2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]\Big)\,\mathrm{coI}[r]+ \\
& \quad 4\,i\,\pi\,\mathrm{rho}\,t\,\mathrm{BesselI}\Big[\frac{1}{2}\times(-2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coI}[2+r]- \\
& \quad 2\,e^{i\,\mathrm{th}}\,\pi\,\mathrm{rho}\,t\,\mathrm{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[r]+ \\
& \quad e^{i\,\mathrm{th}}\,h\,\mathrm{BesselK}\Big[\frac{h-r}{2},\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[r]-e^{i\,\mathrm{th}}\,r\,\mathrm{BesselK}\Big[\frac{h-r}{2},\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[r]- \\
& \quad 2\,e^{i\,\mathrm{th}}\,\pi\,\mathrm{rho}\,t\,\mathrm{BesselK}\Big[\frac{1}{2}\times(2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[r]+ \\
& \quad 4\,i\,\pi\,\mathrm{rho}\,t\,\mathrm{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[2+r]\Big), \\
& -\frac{1}{4\pi\,\mathrm{rho}}\,e^{-i\,\mathrm{th}}\,t^{1+p}\Big(4\,e^{i\,\mathrm{th}}\,\pi\,\mathrm{rho}\,t\,\mathrm{BesselI}\Big[\frac{h-r}{2},\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coI}[r]+ \\
& \quad i\Big(2\pi\,\mathrm{rho}\,t\,\mathrm{BesselI}\Big[\frac{1}{2}\times(-4+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]+(2-h+r)\,\mathrm{BesselI}\Big[\frac{1}{2}\times(-2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]+ \\
& \quad\quad 2\pi\,\mathrm{rho}\,t\,\mathrm{BesselI}\Big[\frac{h-r}{2},\ 2\pi\,\mathrm{rho}\,t\Big]\Big)\,\mathrm{coI}[2+r]+ \\
& \quad 4\,e^{i\,\mathrm{th}}\,\pi\,\mathrm{rho}\,t\,\mathrm{BesselK}\Big[\frac{h-r}{2},\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[r]-2\,i\,\pi\,\mathrm{rho}\,t\,\mathrm{BesselK}\Big[\frac{1}{2}\times(-4+h-r),\ 2\pi\,\mathrm{rho}\,t\Big] \\
& \quad \mathrm{coK}[2+r]+2\,i\,\mathrm{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[2+r]- \\
& \quad i\,h\,\mathrm{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[2+r]+i\,r\,\mathrm{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\ 2\pi\,\mathrm{rho}\,t\Big] \\
& \quad \mathrm{coK}[2+r]-2\,i\,\pi\,\mathrm{rho}\,t\,\mathrm{BesselK}\Big[\frac{h-r}{2},\ 2\pi\,\mathrm{rho}\,t\Big]\,\mathrm{coK}[2+r]\Big)\Big\}
\end{aligned}
$$

Try to get a simplification with the contiguous relations.

In[ ]:=
```
eq1b = eq1a /. Besseldn[1/2 × (2 + h - r)] // Simplify
```

Out[ ]=
$$i\, t^{2+p} \left(e^{i\, th}\, \text{BesselI}\left[\tfrac{1}{2} \times (-2 + h - r),\ 2\, \pi\, \text{rho}\, t\right] \text{coI}[r] + i \left(\text{BesselI}\left[\tfrac{1}{2} \times (-2 + h - r),\ 2\, \pi\, \text{rho}\, t\right] \text{coI}[2 + r] + \text{BesselK}\left[\tfrac{1}{2} \times (-2 + h - r),\ 2\, \pi\, \text{rho}\, t\right] \left(i\, e^{i\, th}\, \text{coK}[r] + \text{coK}[2 + r]\right)\right)\right)$$

In[ ]:=
```
eq2b = eq2a //. Besselup[1/2 × (-4 + h - r)] // Simplify
```

Out[ ]=
$$-e^{-i\, th}\, t^{2+p} \left(e^{i\, th}\, \text{BesselI}\left[\tfrac{h - r}{2},\ 2\, \pi\, \text{rho}\, t\right] \text{coI}[r] + i\, \text{BesselI}\left[\tfrac{h - r}{2},\ 2\, \pi\, \text{rho}\, t\right] \text{coI}[2 + r] + \text{BesselK}\left[\tfrac{h - r}{2},\ 2\, \pi\, \text{rho}\, t\right] \left(e^{i\, th}\, \text{coK}[r] - i\, \text{coK}[2 + r]\right)\right)$$

That is indeed simpler.

In[ ]:=
```
Solve[{eq1b == 0, eq2b == 0}, {coI[r + 2], coK[r + 2]}]
```

Out[ ]=
$$\left\{\left\{\text{coI}[2 + r] \to i\, e^{i\, th}\, \text{coI}[r],\ \text{coK}[2 + r] \to -i\, e^{i\, th}\, \text{coK}[r]\right\}\right\}$$