## 1b. Bruhat coordinates to Iwasawa coordinates

We check the relation in Lemma 2.1 in §2.1.1

In[ ]:=
```
Clear[b, r, t, c, DD]
(* DD== 2 I r + t^2+Abs[b]^2}*)
ba = -c b / (Conjugate[c] ^ 2 DD)
ra = -r Abs[c] ^ (-2) Abs[DD] ^ (-2)
ta = t Abs[c] ^ (-1) Abs[DD] ^ (-1)
```

Out[ ]= $-\frac{\text{b c}}{\text{DD Conjugate[c]}^2}$

Out[ ]= $-\frac{\text{r}}{\text{Abs[c]}^2\ \text{Abs[DD]}^2}$

Out[ ]= $\frac{\text{t}}{\text{Abs[c] Abs[DD]}}$

We want to see that

$a(t')^{-1}\, n(b', r')^{-1}\, w\, h(c)\, n(b, r)\, a(\text{t})$

is equal to k1

In[ ]:=
```
k1a = am[ta ^ (-1)].nm[-ba, -ra].wm.hm[c].nm[b, r]. am[t] //. gensub // Simplify
```

Out[ ]= $\Big\{\Big\{\Big(\text{c}\left((2\text{ r}-i\text{ t}^2)^2+2\times(2\,i\text{ r}+\text{t}^2)\text{ Abs[b]}^2-\text{Abs[b]}^4-\text{Abs[DD]}^2\right)\Big)\Big/\left(2\text{ t}^2\text{ Abs[c] Abs[DD]}\right)+\frac{\text{b Abs[c]}^3\text{ Abs[DD] Conjugate[b]}}{\text{c DD t}^2\text{ Conjugate[c]}^2},\ \frac{\text{b c}\left(-2\,i\text{ r}-\text{t}^2+\text{Abs[b]}^2\right)}{\text{t Abs[c] Abs[DD]}}-\frac{\text{b Abs[c]}^3\text{ Abs[DD]}}{\text{c DD t Conjugate[c]}^2},$

$\frac{\text{c}\left(-4\text{ r}^2-\text{t}^4-4\,i\text{ r Abs[b]}^2+\text{Abs[b]}^4+\text{Abs[DD]}^2\right)}{2\text{ t}^2\text{ Abs[c] Abs[DD]}}-\frac{\text{b Abs[c]}^3\text{ Abs[DD] Conjugate[b]}}{\text{c DD t}^2\text{ Conjugate[c]}^2}\Big\},$

$\{(\text{Conjugate[b]}\,(\text{c}\,(2\,i\text{ r}+\text{t}^2-\text{Abs[b]}^2)\text{ Conjugate[c]}+\text{Abs[c]}^2\text{ Conjugate[DD]}))/(\text{c}^2\text{ t Conjugate[DD]}),$

$(2\text{ b c Conjugate[b] Conjugate[c]}-\text{Abs[c]}^2\text{ Conjugate[DD]})/(\text{c}^2\text{ Conjugate[DD]}),$

$(\text{Conjugate[b]}\,(\text{c}\,(-2\,i\text{ r}+\text{t}^2+\text{Abs[b]}^2)\text{ Conjugate[c]}-\text{Abs[c]}^2\text{ Conjugate[DD]}))/(\text{c}^2\text{ t Conjugate[DD]})\},$

$\Big\{\frac{\text{c}\left(4\text{ r}^2+\text{t}^4+4\,i\text{ r Abs[b]}^2-\text{Abs[b]}^4-\text{Abs[DD]}^2\right)}{2\text{ t}^2\text{ Abs[c] Abs[DD]}}+\frac{\text{b Abs[c]}^3\text{ Abs[DD] Conjugate[b]}}{\text{c DD t}^2\text{ Conjugate[c]}^2},$

$\frac{\text{b c}\left(-2\,i\text{ r}+\text{t}^2+\text{Abs[b]}^2\right)}{\text{t Abs[c] Abs[DD]}}-\frac{\text{b Abs[c]}^3\text{ Abs[DD]}}{\text{c DD t Conjugate[c]}^2},$

$\Big(\text{c}\left(-(2\text{ r}+i\text{ t}^2)^2+2\times(-2\,i\text{ r}+\text{t}^2)\text{ Abs[b]}^2+\text{Abs[b]}^4+\text{Abs[DD]}^2\right)\Big)\Big/\left(2\text{ t}^2\text{ Abs[c] Abs[DD]}\right)-\frac{\text{b Abs[c]}^3\text{ Abs[DD] Conjugate[b]}}{\text{c DD t}^2\text{ Conjugate[c]}^2}\Big\}\Big\}$

k1 should have zeros at several positions. Let us first try to arrange that, by suitable substitutions.

In[ ]:=
```
sub1 = {-2 I r + t^2 + Abs[b]^2 → Conjugate[DD], Abs[c]^2 → c Conjugate[c]}
```

Out[ ]=
$$\left\{-2\, i\, r + t^2 + \text{Abs}[b]^2 \to \text{Conjugate}[DD],\ \text{Abs}[c]^2 \to c\ \text{Conjugate}[c]\right\}$$

In[ ]:=
```
k1a[[2, 3]] /. sub1
```

Out[ ]=
$$0$$

In[ ]:=
```
k1b = k1a //. sub1 // Simplify
```

Out[ ]=
$$\Bigg\{\Bigg\{\left(c\left(\left(2\, r - i\, t^2\right)^2 + 2 \times \left(2\, i\, r + t^2\right) \text{Abs}[b]^2 - \text{Abs}[b]^4 - \text{Abs}[DD]^2\right)\right) \Big/ \left(2\, t^2\, \text{Abs}[c]\, \text{Abs}[DD]\right) + \frac{b\, \text{Abs}[c]^3\, \text{Abs}[DD]\, \text{Conjugate}[b]}{c\, DD\, t^2\, \text{Conjugate}[c]^2},\ \frac{b\, c\left(-2\, i\, r - t^2 + \text{Abs}[b]^2\right)}{t\, \text{Abs}[c]\, \text{Abs}[DD]} - \frac{b\, \text{Abs}[c]^3\, \text{Abs}[DD]}{c\, DD\, t\, \text{Conjugate}[c]^2},$$
$$\frac{c\left(-4\, r^2 - t^4 - 4\, i\, r\, \text{Abs}[b]^2 + \text{Abs}[b]^4 + \text{Abs}[DD]^2\right)}{2\, t^2\, \text{Abs}[c]\, \text{Abs}[DD]} - \frac{b\, \text{Abs}[c]^3\, \text{Abs}[DD]\, \text{Conjugate}[b]}{c\, DD\, t^2\, \text{Conjugate}[c]^2}\Bigg\},$$
$$\Bigg\{\left(\text{Conjugate}[b]\, \text{Conjugate}[c]\left(2\, i\, r + t^2 - \text{Abs}[b]^2 + \text{Conjugate}[DD]\right)\right) / \left(c\, t\, \text{Conjugate}[DD]\right),\ \frac{\text{Conjugate}[c]\left(2\, b\, \text{Conjugate}[b] - \text{Conjugate}[DD]\right)}{c\, \text{Conjugate}[DD]},\ 0\Bigg\},$$
$$\Bigg\{\frac{c\left(4\, r^2 + t^4 + 4\, i\, r\, \text{Abs}[b]^2 - \text{Abs}[b]^4 - \text{Abs}[DD]^2\right)}{2\, t^2\, \text{Abs}[c]\, \text{Abs}[DD]} + \frac{b\, \text{Abs}[c]^3\, \text{Abs}[DD]\, \text{Conjugate}[b]}{c\, DD\, t^2\, \text{Conjugate}[c]^2},$$
$$-\frac{b\, \text{Abs}[c]^3\, \text{Abs}[DD]}{c\, DD\, t\, \text{Conjugate}[c]^2} + \frac{b\, c\, \text{Conjugate}[DD]}{t\, \text{Abs}[c]\, \text{Abs}[DD]},$$
$$\left(c\left(-\left(2\, r + i\, t^2\right)^2 + 2 \times \left(-2\, i\, r + t^2\right) \text{Abs}[b]^2 + \text{Abs}[b]^4 + \text{Abs}[DD]^2\right)\right) \Big/ \left(2\, t^2\, \text{Abs}[c]\, \text{Abs}[DD]\right) - \frac{b\, \text{Abs}[c]^3\, \text{Abs}[DD]\, \text{Conjugate}[b]}{c\, DD\, t^2\, \text{Conjugate}[c]^2}\Bigg\}\Bigg\}$$

In[ ]:=
```
sub2 = {r Abs[b]^2 → Abs[b]^2 (-Conjugate[DD] + t^2 + Abs[b]^2) / (2 I),
    r^2 → (Abs[DD]^2 - (t^2 + Abs[b]^2)^2) / 4, Conjugate[DD] → Abs[DD]^2 / DD};
sub2a = {c^2 Conjugate[c]^2 → Abs[c]^4, b Conjugate[b] → Abs[b]^2};
```

In[ ]:=
```
k1b[[1, 3]] //. sub2 // Simplify
% /. sub2a // Simplify
```

Out[ ]=
$$\left(\text{Abs}[DD]\left(-b\, \text{Abs}[c]^4\, \text{Conjugate}[b] + c^2\, \text{Abs}[b]^2\, \text{Conjugate}[c]^2\right)\right) / \left(c\, DD\, t^2\, \text{Abs}[c]\, \text{Conjugate}[c]^2\right)$$

Out[ ]=
$$0$$

```
k1b //. sub2 // Simplify
k1c = % //. sub2a // Simplify
```

$$\Big\{\Big\{\big(c\,((2\,r - i\,t^2)^2 + 2\times(2\,i\,r + t^2)\,\text{Abs}[b]^2 - \text{Abs}[b]^4 - \text{Abs}[DD]^2)\big)\big/\big(2\,t^2\,\text{Abs}[c]\,\text{Abs}[DD]\big) + \frac{b\,\text{Abs}[c]^3\,\text{Abs}[DD]\,\text{Conjugate}[b]}{c\,DD\,t^2\,\text{Conjugate}[c]^2},\ \frac{b\,c\,(-2\,i\,r - t^2 + \text{Abs}[b]^2)}{t\,\text{Abs}[c]\,\text{Abs}[DD]} - \frac{b\,\text{Abs}[c]^3\,\text{Abs}[DD]}{c\,DD\,t\,\text{Conjugate}[c]^2},$$
$$\big(\text{Abs}[DD]\,(-b\,\text{Abs}[c]^4\,\text{Conjugate}[b] + c^2\,\text{Abs}[b]^2\,\text{Conjugate}[c]^2)\big)\big/\big(c\,DD\,t^2\,\text{Abs}[c]\,\text{Conjugate}[c]^2\big)\Big\},$$
$$\Big\{\frac{1}{c\,t\,\text{Abs}[DD]^2}\big(DD\,(2\,i\,r + t^2) - DD\,\text{Abs}[b]^2 + \text{Abs}[DD]^2\big)\,\text{Conjugate}[b]\,\text{Conjugate}[c],\ -\frac{(\text{Abs}[DD]^2 - 2\,b\,DD\,\text{Conjugate}[b])\,\text{Conjugate}[c]}{c\,\text{Abs}[DD]^2},\ 0\Big\},$$
$$\Big\{\big(\text{Abs}[DD]\,(b\,\text{Abs}[c]^4\,\text{Conjugate}[b] - c^2\,\text{Abs}[b]^2\,\text{Conjugate}[c]^2)\big)\big/\big(c\,DD\,t^2\,\text{Abs}[c]\,\text{Conjugate}[c]^2\big),\ \frac{b\,\text{Abs}[DD]\,(-\text{Abs}[c]^4 + c^2\,\text{Conjugate}[c]^2)}{c\,DD\,t\,\text{Abs}[c]\,\text{Conjugate}[c]^2},$$
$$\big(c\,(-(2\,r + i\,t^2)^2 + 2\times(-2\,i\,r + t^2)\,\text{Abs}[b]^2 + \text{Abs}[b]^4 + \text{Abs}[DD]^2)\big)\big/\big(2\,t^2\,\text{Abs}[c]\,\text{Abs}[DD]\big) - \frac{b\,\text{Abs}[c]^3\,\text{Abs}[DD]\,\text{Conjugate}[b]}{c\,DD\,t^2\,\text{Conjugate}[c]^2}\Big\}\Big\}$$

$$\Big\{\Big\{\big(c\,((2\,r - i\,t^2)^2 + 2\times(2\,i\,r + t^2)\,\text{Abs}[b]^2 - \text{Abs}[b]^4 - \text{Abs}[DD]^2)\big)\big/\big(2\,t^2\,\text{Abs}[c]\,\text{Abs}[DD]\big) + \frac{\text{Abs}[b]^2\,\text{Abs}[c]^3\,\text{Abs}[DD]}{c\,DD\,t^2\,\text{Conjugate}[c]^2},\ \frac{b\,c\,(-2\,i\,r - t^2 + \text{Abs}[b]^2)}{t\,\text{Abs}[c]\,\text{Abs}[DD]} - \frac{b\,\text{Abs}[c]^3\,\text{Abs}[DD]}{c\,DD\,t\,\text{Conjugate}[c]^2},\ 0\Big\},$$
$$\Big\{\frac{1}{c\,t\,\text{Abs}[DD]^2}\big(DD\,(2\,i\,r + t^2) - DD\,\text{Abs}[b]^2 + \text{Abs}[DD]^2\big)\,\text{Conjugate}[b]\,\text{Conjugate}[c],\ -\frac{(-2\,DD\,\text{Abs}[b]^2 + \text{Abs}[DD]^2)\,\text{Conjugate}[c]}{c\,\text{Abs}[DD]^2},\ 0\Big\},$$
$$\Big\{0,\ 0,\ \big(c\,(-(2\,r + i\,t^2)^2 + 2\times(-2\,i\,r + t^2)\,\text{Abs}[b]^2 + \text{Abs}[b]^4 + \text{Abs}[DD]^2)\big)\big/\big(2\,t^2\,\text{Abs}[c]\,\text{Abs}[DD]\big) - \frac{\text{Abs}[b]^2\,\text{Abs}[c]^3\,\text{Abs}[DD]}{c\,DD\,t^2\,\text{Conjugate}[c]^2}\Big\}\Big\}$$

```
sub3 = {r → (DD − Conjugate [DD]) / (4 I), Abs[b] ^ 2 → (DD + Conjugate [DD]) / 2 − t ^ 2,
    Abs[b] ^ 4 → ((DD + Conjugate [DD]) / 2 − t ^ 2) ^ 2, Abs[c] ^ 3 → c ^ 2 Conjugate [c] ^ 2 / Abs[c]};
sub3a = {Abs[DD] ^ 2 → DD Conjugate [DD]};
```

The following element should be equal to
c Conjugate[DD]/Abs[c]Abs[DD]

```
In[ ]:= k1c[[3, 3]]
        % //. sub3 // Simplify
        % //. sub3a // Simplify
```

Out[ ]= $\left(c\left(-\left(2 r+i t^{2}\right)^{2}+2 \times\left(-2 i r+t^{2}\right) \text{Abs}[b]^{2}+\text{Abs}[b]^{4}+\text{Abs}[DD]^{2}\right)\right) / \left(2 t^{2} \text{Abs}[c] \text{Abs}[DD]\right)-\frac{\text{Abs}[b]^{2} \text{Abs}[c]^{3} \text{Abs}[DD]}{c\, DD\, t^{2} \text{Conjugate}[c]^{2}}$

Out[ ]= $\left(c\left(\text{Abs}[DD]^{2}\left(2 t^{2}-\text{Conjugate}[DD]\right)+DD\, \text{Conjugate}[DD]^{2}\right)\right) / \left(2\, DD\, t^{2} \text{Abs}[c] \text{Abs}[DD]\right)$

Out[ ]= $\frac{c\, \text{Conjugate}[DD]}{\text{Abs}[c] \text{Abs}[DD]}$

Now the third column has the desired form.

```
In[ ]:= k1c //. sub3 // Simplify ;
        k1d = % //. sub3a // Simplify
```

Out[ ]= $\left\{\left\{\frac{c\left(-2 t^{2}+\text{Conjugate}[DD]\right)}{\text{Abs}[c] \text{Abs}[DD]},-\frac{2 b c t}{\text{Abs}[c] \text{Abs}[DD]}, 0\right\},\left\{\frac{2\, DD\, t\, \text{Conjugate}[b] \text{Conjugate}[c]}{c\, \text{Abs}[DD]^{2}}, \frac{DD\left(DD-2 t^{2}\right) \text{Conjugate}[c]}{c\, \text{Abs}[DD]^{2}}, 0\right\},\left\{0,0, \frac{c\, \text{Conjugate}[DD]}{\text{Abs}[c] \text{Abs}[DD]}\right\}\right\}$

Also the third row has the right form :

```
In[ ]:= k1d //. {DD Abs[DD]^(-2) → Conjugate[DD]^(-1)} // Simplify // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{c\left(-2 t^{2}+\text{Conjugate}[DD]\right)}{\text{Abs}[c] \text{Abs}[DD]} & -\frac{2 b c t}{\text{Abs}[c] \text{Abs}[DD]} & 0 \\ \frac{2 t\, \text{Conjugate}[b] \text{Conjugate}[c]}{c\, \text{Conjugate}[DD]} & \frac{\left(DD-2 t^{2}\right) \text{Conjugate}[c]}{c\, \text{Conjugate}[DD]} & 0 \\ 0 & 0 & \frac{c\, \text{Conjugate}[DD]}{\text{Abs}[c] \text{Abs}[DD]} \end{pmatrix}$$

This is the element $k_1$ in the lemma .

This element is in K⊂G. Let us check that directly.

```
In[ ]:= Det[k1d] //. {zz_ Conjugate[zz_] :→ Abs[zz]^2, t^2 → (DD + Conjugate[DD]) / 2 - Abs[b]^2,
           t^4 → ((DD + Conjugate[DD]) / 2 - Abs[b]^2)^2} // Simplify
        % /. Conjugate[xx_] :→ Abs[xx]^2 / xx // Simplify
```

Out[ ]= $\frac{c\, DD^{2} \text{Conjugate}[c] \text{Conjugate}[DD]^{2}}{\text{Abs}[c]^{2} \text{Abs}[DD]^{4}}$

Out[ ]= 1

In[ ]:=
```
inG[k1d] //. {Conjugate[t] → t, Abs[t] → t, t^2 → (DD + Conjugate[DD]) / 2 - Abs[b]^2,
    t^4 → ((DD + Conjugate[DD]) / 2 - Abs[b]^2)^2} // Simplify
% //. {zz_ Conjugate[zz_] :→ Abs[zz]^2, zz_^ee_ Conjugate[zz_]^ee_ :→ Abs[zz]^(2 ee)} //
  Simplify
```

Out[ ]=
$$\frac{\text{DD Conjugate[DD]}}{\text{Abs[DD]}^2} == 1 \ \&\& \ \frac{\text{c DD}^2 \text{ Conjugate[c] Conjugate[DD]}^2}{\text{Abs[c]}^2 \text{ Abs[DD]}^4} == 1$$

Out[ ]= True

In[ ]:=