20b. More complicated cases

In these cases more components are needed.

```
In[ ]:= Clear[m0, r0, wh, factor]
whsub = {wh^(0,0,2)[kp_, s_, tau_] :> (1/4 - kp/tau + (s^2 - 1/4) tau^(-2)) wh[kp, s, tau]};
```

First upward shift operator, $\varepsilon=1$, $p \geq m_0$, $-p \leq r_0 < p$

```
In[ ]:= F = tht[0] × f[r0, t] × Phi[h, p, r0, p] + tht[1] × f[r0 + 2, t] × Phi[h, p, r0 + 2, p]
Out[ ]= f[r0, t] × Phi[h, p, r0, p] × tht[0] + f[2 + r0, t] × Phi[h, p, 2 + r0, p] × tht[1]
```

Determine component of order $r_0 + 2$ of F with use of the fact that $(S^3)_{-1} F = 0$.

```
In[ ]:= Coefficient[sh[3, -1, F, subnab], Phi[h + 3, p - 1, r0 + 1, p - 1]] /. eps → 1 // Simplify
sol = Solve[% == 0, f[r0 + 2, t]][[1]] // Simplify
```

Out[ ]= $-\frac{1}{4 \times (1+p)} \, p\, \mathrm{tht}[0] \left((4 - h + 2p + r0 - 4\, \mathrm{ell}\, \pi t^2)\, f[r0, t] + 4 i \sqrt{2\pi}\, t \sqrt{\mathrm{Abs}[\mathrm{ell}]}\, f[2 + r0, t] - 2 t f^{(0,1)}[r0, t] \right)$

Out[ ]= $\left\{ f[2 + r0, t] \to -\left(\left(i \left((-4 + h - 2p - r0 + 4\, \mathrm{ell}\, \pi t^2)\, f[r0, t] + 2 t f^{(0,1)}[r0, t] \right) \right) \Big/ \left(4 \sqrt{2\pi}\, t \sqrt{\mathrm{Abs}[\mathrm{ell}]} \right) \right) \right\}$

Now we need two Whittaker functions

```
In[ ]:= fr0 = t^(m0 + 1) wh[kap, nu/2, 2 Pi Abs[ell] t^2]
fr1 = t^(m0 + 1) wh[kap - 1, nu/2, 2 Pi Abs[ell] t^2]
```

Out[ ]= $t^{1+m0}\, \mathrm{wh}\left[\mathrm{kap}, \frac{\mathrm{nu}}{2}, 2 \pi t^2 \mathrm{Abs}[\mathrm{ell}]\right]$

Out[ ]= $t^{1+m0}\, \mathrm{wh}\left[-1 + \mathrm{kap}, \frac{\mathrm{nu}}{2}, 2 \pi t^2 \mathrm{Abs}[\mathrm{ell}]\right]$

In[ ]:=
```
F /. sol // Simplify (* the expression for f_{2+r_0} inserted *)
sh[3, 1, %, subnab] /. eps → 1 /. Abs[ell] → ell
(* carry out upward shift operator and determine its lowest component FP1*)
FP1 = Coefficient[%, Phi[3 + h, 1 + p, 1 + r0, 1 + p]] /. f[r0, t] → fr0 /.
    f^(0,ee_)[r0, t] :→ D[fr0, {t, ee}] /. whsub // Simplify
```

Out[ ]=
$$\mathrm{f}[\mathrm{r0},\mathrm{t}]\times\mathrm{Phi}[\mathrm{h},\mathrm{p},\mathrm{r0},\mathrm{p}]\times\mathrm{tht}[0]-\Big(i\,\mathrm{Phi}[\mathrm{h},\mathrm{p},2+\mathrm{r0},\mathrm{p}]\times\mathrm{tht}[1]\left(\left(-4+\mathrm{h}-2\,\mathrm{p}-\mathrm{r0}+4\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right)\mathrm{f}[\mathrm{r0},\mathrm{t}]+2\,\mathrm{t}\,\mathrm{f}^{(0,1)}[\mathrm{r0},\mathrm{t}]\right)\Big)\Big/\left(4\sqrt{2\pi}\,\mathrm{t}\sqrt{\mathrm{Abs}[\mathrm{ell}]}\right)$$

Out[ ]=
$$\Big(4\sqrt{\mathrm{ell}}\,\sqrt{\pi}\,\mathrm{t}\,\mathrm{Phi}[3+\mathrm{h},1+\mathrm{p},1+\mathrm{r0},1+\mathrm{p}]\times\big(\mathrm{f}[\mathrm{r0},\mathrm{t}]\left(\mathrm{h}\,(4+4\,\mathrm{p})\,\mathrm{tht}[0]+4\,\mathrm{r0}\,\mathrm{tht}[0]+16\,\mathrm{ell}\,\pi\,\mathrm{t}^2\,\mathrm{tht}[0]+\mathrm{p}\left(4\,\mathrm{r0}+16\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right)\mathrm{tht}[0]\right)+2\times(4+4\,\mathrm{p})\,\mathrm{t}\,\mathrm{tht}[0]\,\mathrm{f}^{(0,1)}[\mathrm{r0},\mathrm{t}]\big)-i\sqrt{2}\,(4+\mathrm{p}+\mathrm{r0})\,\mathrm{Phi}[3+\mathrm{h},1+\mathrm{p},3+\mathrm{r0},1+\mathrm{p}]\times\mathrm{tht}[1]\Big(\left(16+\mathrm{h}^2-4\,\mathrm{p}^2+8\,\mathrm{r0}+\mathrm{r0}^2-16\,\mathrm{ell}\,\pi\,\mathrm{t}^2-8\,\mathrm{ell}\,\pi\,\mathrm{r0}\,\mathrm{t}^2+16\,\mathrm{ell}^2\,\pi^2\,\mathrm{t}^4-2\,\mathrm{h}\left(4+\mathrm{r0}-4\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right)\right)\mathrm{f}[\mathrm{r0},\mathrm{t}]+4\,\mathrm{t}\left(\left(-3+\mathrm{h}-\mathrm{r0}+4\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right)\mathrm{f}^{(0,1)}[\mathrm{r0},\mathrm{t}]+\mathrm{t}\,\mathrm{f}^{(0,2)}[\mathrm{r0},\mathrm{t}]\right)\Big)\Big)\Big/\left(64\sqrt{\mathrm{ell}}\,(1+\mathrm{p})\sqrt{\pi}\,\mathrm{t}\right)$$

Out[ ]=
$$\frac{1}{4}\,\mathrm{t}^{1+\mathrm{m0}}\,\mathrm{tht}[0]\left(\left(2+\mathrm{h}+2\,\mathrm{m0}+\mathrm{r0}+4\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right)\mathrm{wh}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\,\pi\,\mathrm{t}^2\,\mathrm{Abs}[\mathrm{ell}]\right]+8\,\pi\,\mathrm{t}^2\,\mathrm{Abs}[\mathrm{ell}]\,\mathrm{wh}^{(0,0,1)}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\,\pi\,\mathrm{t}^2\,\mathrm{Abs}[\mathrm{ell}]\right]\right)$$

We want to determine the ratio with the highest component of F

In[ ]:=
```
eqn = FP1 - factor tht[0] fr1 //.
   {Abs[ell] → ell, h → 2 j + 3 p, r0 → p - 2 m0, p → -kap - (j + 1) / 2} // Simplify
```

Out[ ]=
$$\mathrm{t}^{1+\mathrm{m0}}\,\mathrm{tht}[0]\left(-\mathrm{factor}\,\mathrm{wh}\left[-1+\mathrm{kap},\frac{\mathrm{nu}}{2},2\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right]+\left(-\mathrm{kap}+\mathrm{ell}\,\pi\,\mathrm{t}^2\right)\mathrm{wh}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right]+2\,\mathrm{ell}\,\pi\,\mathrm{t}^2\,\mathrm{wh}^{(0,0,1)}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right]\right)$$

The Whittaker function may be any of the three standard Whittaker functions. We use substitutions defined in A2d (appendix).

In[ ]:=
```
eqn /. wh → WhittakerW /. Whdn[kap + 1](* relations for Whittaker functions ,
  to simplify the expression *) /. kap → -p - (j + 1) / 2
  (* value of κ in Table 19 *) // Simplify
Solve[% == 0, factor][[1]] // Factor
```

Out[ ]=
$$-\frac{1}{4}\times\left(-4+4\,\mathrm{factor}-\mathrm{j}^2+\mathrm{nu}^2-8\,\mathrm{p}-4\,\mathrm{p}^2-4\,\mathrm{j}\,(1+\mathrm{p})\right)\mathrm{t}^{1+\mathrm{m0}}\,\mathrm{tht}[0]\,\mathrm{WhittakerW}\left[-\frac{3}{2}-\frac{\mathrm{j}}{2}-\mathrm{p},\frac{\mathrm{nu}}{2},2\,\mathrm{ell}\,\pi\,\mathrm{t}^2\right]$$

Out[ ]=
$$\left\{\mathrm{factor}\to\frac{1}{4}\times(2+\mathrm{j}-\mathrm{nu}+2\,\mathrm{p})\times(2+\mathrm{j}+\mathrm{nu}+2\,\mathrm{p})\right\}$$

```
In[ ]:= eqn /. wh → WhittakerV /. Whrel /. Whdn[kap + 1] /. kap → -p - (j + 1) / 2 // Simplify
        Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\left((1+\text{factor})\, t^{1+m0}\, \text{tht}[0] \times \text{WhittakerV}\left[-\frac{3}{2}-\frac{j}{2}-p,\ \frac{nu}{2},\ 2\,\text{ell}\,\pi\, t^2\right]\right)$

Out[ ]= $\{\text{factor} \to -1\}$

```
In[ ]:= eqn /. wh → WhittakerM /. Whdn[kap + 1] /. kap → -p - (j + 1) / 2 // Simplify
        Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\frac{1}{2} \times (-2+2\,\text{factor}-j-nu-2\,p)\, t^{1+m0}\, \text{tht}[0]\, \text{WhittakerM}\left[-\frac{3}{2}-\frac{j}{2}-p,\ \frac{nu}{2},\ 2\,\text{ell}\,\pi\, t^2\right]$

Out[ ]= $\left\{\text{factor} \to \frac{1}{2} \times (2+j+nu+2\,p)\right\}$

These three factors are to be used in Table 4.8 (third box on the right).

First upward shift operator, ε=1, $p \geq m_0, r_0 = p$

Now there is only one component

```
In[ ]:= F = tht[0] × f[p, t] × Phi[h, p, p, p]
```

Out[ ]= f[p, t] × Phi[h, p, p, p] × tht[0]

We need two Whittaker functions

```
In[ ]:= fr0 = t ^ (m0 + 1) wh[kap, nu / 2, 2 Pi Abs[ell] t ^ 2]
        fr1 = t ^ (m0 + 1) wh[kap - 1, nu / 2, 2 Pi Abs[ell] t ^ 2]
```

Out[ ]= $t^{1+m0}\, \text{wh}\left[\text{kap},\ \frac{nu}{2},\ 2\,\pi\, t^2\, \text{Abs}[\text{ell}]\right]$

Out[ ]= $t^{1+m0}\, \text{wh}\left[-1+\text{kap},\ \frac{nu}{2},\ 2\,\pi\, t^2\, \text{Abs}[\text{ell}]\right]$

```
In[ ]:=
        sh[3, 1, F, subnab] /. eps → 1
        FP1 = Coefficient[%, Phi[3 + h, 1 + p, 1 + p, 1 + p]] /. f[p, t] → fr0 /.
            f^(0,ee_)[p, t] :→ D[fr0, {t, ee}] /. whsub // Simplify
```

Out[ ]= $\frac{1}{4}\left(\left(h+p+4\,\text{ell}\,\pi\, t^2\right) f[p,t] \times \text{Phi}[3+h,\ 1+p,\ 1+p,\ 1+p] \times \text{tht}[0] + 2\, t\, \text{Phi}[3+h,\ 1+p,\ 1+p,\ 1+p] \times \text{tht}[0]\, f^{(0,1)}[p,t]\right)$

Out[ ]= $\frac{1}{4}\, t^{1+m0}\, \text{tht}[0]\left(\left(2+h+2\,m0+p+4\,\text{ell}\,\pi\, t^2\right) \text{wh}\left[\text{kap},\ \frac{nu}{2},\ 2\,\pi\, t^2\, \text{Abs}[\text{ell}]\right] + 8\,\pi\, t^2\, \text{Abs}[\text{ell}]\, \text{wh}^{(0,0,1)}\left[\text{kap},\ \frac{nu}{2},\ 2\,\pi\, t^2\, \text{Abs}[\text{ell}]\right]\right)$

In[ ]:=
```
eqn = FP1 - factor tht[0] fr1 //.
    {Abs[ell] → ell, h → 2 j + 3 p, p → -kap - (j + 1) / 2, m0 → 0} // Simplify
```

Out[ ]= $t\ \text{tht}[0]\left(-\text{factor wh}\left[-1+\text{kap}, \frac{\text{nu}}{2}, 2\ \text{ell}\ \pi\ t^2\right]+\left(-\text{kap}+\text{ell}\ \pi\ t^2\right)\text{wh}\left[\text{kap}, \frac{\text{nu}}{2}, 2\ \text{ell}\ \pi\ t^2\right]+2\ \text{ell}\ \pi\ t^2\ \text{wh}^{(0,0,1)}\left[\text{kap}, \frac{\text{nu}}{2}, 2\ \text{ell}\ \pi\ t^2\right]\right)$

The three cases :

In[ ]:=
```
eqn /. wh → WhittakerW /. Whdn[kap + 1] /. kap → -p - (j + 1) / 2 // Simplify
Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\frac{1}{4}\times\left(-4+4\ \text{factor}-j^2+\text{nu}^2-8\ p-4\ p^2-4\ j\ (1+p)\right)\ t\ \text{tht}[0]\ \text{WhittakerW}\left[-\frac{3}{2}-\frac{j}{2}-p, \frac{\text{nu}}{2}, 2\ \text{ell}\ \pi\ t^2\right]$

Out[ ]= $\left\{\text{factor}\to\frac{1}{4}\times(2+j-\text{nu}+2\ p)\times(2+j+\text{nu}+2\ p)\right\}$

In[ ]:=
```
eqn /. wh → WhittakerV /. Whrel /. Whdn[kap + 1] /. kap → -p - (j + 1) / 2 // Simplify
Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\left((1+\text{factor})\ t\ \text{tht}[0]\times\text{WhittakerV}\left[-\frac{3}{2}-\frac{j}{2}-p, \frac{\text{nu}}{2}, 2\ \text{ell}\ \pi\ t^2\right]\right)$

Out[ ]= {factor → -1}

In[ ]:=
```
eqn /. wh → WhittakerM /. Whdn[kap + 1] /. kap → -p - (j + 1) / 2 // Simplify
Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\frac{1}{2}\times(-2+2\ \text{factor}-j-\text{nu}-2\ p)\ t\ \text{tht}[0]\ \text{WhittakerM}\left[-\frac{3}{2}-\frac{j}{2}-p, \frac{\text{nu}}{2}, 2\ \text{ell}\ \pi\ t^2\right]$

Out[ ]= $\left\{\text{factor}\to\frac{1}{2}\times(2+j+\text{nu}+2\ p)\right\}$

Also in third box on the right in Table 4.8.

In[ ]:=

Second upward shift operator, ε=-1, $p \geq m_0, r_0 < p$

In[ ]:=
```
F = tht[0] × f[r0, t] × Phi[h, p, r0, p] + tht[1] × f[r0 - 2, t] × Phi[h, p, r0 - 2, p]
```

Out[ ]= f[r0, t] × Phi[h, p, r0, p] × tht[0] + f[-2 + r0, t] × Phi[h, p, -2 + r0, p] × tht[1]

Determine component of order $r_0 - 2$

In[ ]:=
```
Coefficient[sh[-3, -1, F, subnab], Phi[h - 3, p - 1, r0 - 1, p - 1]] /. eps → -1 // Simplify
sol = Solve[% == 0, f[r0 - 2, t]][[1]] // Simplify
```

Out[ ]= $-\frac{1}{4\times(1+p)} p\,\mathrm{tht}[0]\left(4\,i\,\sqrt{2\pi}\,t\,\sqrt{\mathrm{Abs[ell]}}\,f[-2+r0,t]+\left(4+h+2p-r0+4\,\mathrm{ell}\,\pi t^2\right)f[r0,t]-2\,t\,f^{(0,1)}[r0,t]\right)$

Out[ ]= $\left\{f[-2+r0,t]\to\left(i\left(\left(4+h+2p-r0+4\,\mathrm{ell}\,\pi t^2\right)f[r0,t]-2\,t\,f^{(0,1)}[r0,t]\right)\right)\Big/\left(4\sqrt{2\pi}\,t\,\sqrt{\mathrm{Abs[ell]}}\right)\right\}$

Now we need two Whittaker functions

In[ ]:=
```
fr0 = t ^ (m0 + 1) wh[kap, nu / 2, 2 Pi Abs[ell] t ^ 2]
fr1 = t ^ (m0 + 1) wh[kap - 1, nu / 2, 2 Pi Abs[ell] t ^ 2]
```

Out[ ]= $t^{1+m0}\,\mathrm{wh}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\pi t^2\,\mathrm{Abs[ell]}\right]$

Out[ ]= $t^{1+m0}\,\mathrm{wh}\left[-1+\mathrm{kap},\frac{\mathrm{nu}}{2},2\pi t^2\,\mathrm{Abs[ell]}\right]$

In[ ]:=
```
F /. sol // Simplify
sh[-3, 1, %, subnab] /. eps → -1
FP1 = Coefficient[%, Phi[-3 + h, 1 + p, -1 + r0, 1 + p]] /. f[r0, t] → fr0 /.
    f^(0,ee_)[r0, t] :→ D[fr0, {t, ee}] /. whsub // Simplify
```

Out[ ]= $f[r0,t]\times\mathrm{Phi}[h,p,r0,p]\times\mathrm{tht}[0]+\left(i\,\mathrm{Phi}[h,p,-2+r0,p]\times\mathrm{tht}[1]\left(\left(4+h+2p-r0+4\,\mathrm{ell}\,\pi t^2\right)f[r0,t]-2\,t\,f^{(0,1)}[r0,t]\right)\right)\Big/\left(4\sqrt{2\pi}\,t\,\sqrt{\mathrm{Abs[ell]}}\right)$

Out[ ]= $\Big(4\sqrt{\pi}\,t\,\sqrt{\mathrm{Abs[ell]}}\,\mathrm{Phi}[-3+h,1+p,-1+r0,1+p]\times\left(f[r0,t]\left(h(-4-4p)\,\mathrm{tht}[0]-4\,r0\,\mathrm{tht}[0]-16\,\mathrm{ell}\,\pi t^2\,\mathrm{tht}[0]+p\left(-4\,r0-16\,\mathrm{ell}\,\pi t^2\right)\mathrm{tht}[0]\right)-2\times(-4-4p)\,t\,\mathrm{tht}[0]\,f^{(0,1)}[r0,t]\right)-i\sqrt{2}\,(4+p-r0)\,\mathrm{Phi}[-3+h,1+p,-3+r0,1+p]\times\mathrm{tht}[1]\Big(\left(16+h^2-4p^2-8\,r0+r0^2+16\,\mathrm{ell}\,\pi t^2-8\,\mathrm{ell}\,\pi\,r0\,t^2+16\,\mathrm{ell}^2\pi^2t^4+h\left(8-2\,r0+8\,\mathrm{ell}\,\pi t^2\right)\right)f[r0,t]-4\,t\left(\left(3+h-r0+4\,\mathrm{ell}\,\pi t^2\right)f^{(0,1)}[r0,t]-t\,f^{(0,2)}[r0,t]\right)\Big)\Big)\Big/\left(64\times(1+p)\sqrt{\pi}\,t\,\sqrt{\mathrm{Abs[ell]}}\right)$

Out[ ]= $-\frac{1}{4}t^{1+m0}\,\mathrm{tht}[0]\left(\left(-2+h-2\,m0+r0+4\,\mathrm{ell}\,\pi t^2\right)\mathrm{wh}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\pi t^2\,\mathrm{Abs[ell]}\right]-8\pi t^2\,\mathrm{Abs[ell]}\,\mathrm{wh}^{(0,0,1)}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\pi t^2\,\mathrm{Abs[ell]}\right]\right)$

In[ ]:=
```
eqn = FP1 - factor tht[0] fr1 //.
    {Abs[ell] → -ell, h → 2 j - 3 p, r0 → -p + 2 m0, p → -kap + (j - 1) / 2} // Simplify
```

Out[ ]= $-t^{1+m0}\,\mathrm{tht}[0]\left(\mathrm{factor}\,\mathrm{wh}\left[-1+\mathrm{kap},\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi t^2\right]+\left(\mathrm{kap}+\mathrm{ell}\,\pi t^2\right)\mathrm{wh}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi t^2\right]+2\,\mathrm{ell}\,\pi t^2\,\mathrm{wh}^{(0,0,1)}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi t^2\right]\right)$

The three cases :

```
In[ ]:= eqn /. wh → WhittakerW /. Whdn[kap + 1] /. kap → -p + (j - 1) / 2 // Simplify
        Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\frac{1}{4}\times(-4+4\,\text{factor}-j^2+nu^2-8\,p-4\,p^2+4\,j\,(1+p))\ t^{1+m0}\ \text{tht}[0]\ \text{WhittakerW}\left[\frac{1}{2}\times(-3+j-2\,p),\ \frac{nu}{2},\ -2\,\text{ell}\,\pi\,t^2\right]$

Out[ ]= $\left\{\text{factor}\to\frac{1}{4}\times(-2+j-nu-2\,p)\times(-2+j+nu-2\,p)\right\}$

```
In[ ]:= eqn /. wh → WhittakerV /. Whrel /. Whdn[kap + 1] /. kap → -p + (j - 1) / 2 // Simplify
        Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\left((1+\text{factor})\ t^{1+m0}\ \text{tht}[0]\times\text{WhittakerV}\left[\frac{1}{2}\times(-3+j-2\,p),\ \frac{nu}{2},\ -2\,\text{ell}\,\pi\,t^2\right]\right)$

Out[ ]= $\{\text{factor}\to-1\}$

```
In[ ]:= eqn /. wh → WhittakerM /. Whdn[kap + 1] /. kap → -p + (j - 1) / 2 // Simplify
        Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\frac{1}{2}\times(-2+2\,\text{factor}+j-nu-2\,p)\ t^{1+m0}\ \text{tht}[0]\ \text{WhittakerM}\left[\frac{1}{2}\times(-3+j-2\,p),\ \frac{nu}{2},\ -2\,\text{ell}\,\pi\,t^2\right]$

Out[ ]= $\left\{\text{factor}\to\frac{1}{2}\times(2-j+nu+2\,p)\right\}$

This goes into the 8-th box on the right in Table 4.8

Second upward shift operator, ε=-1, $p \geq m_0, r_0 = -p$

```
In[ ]:= F = tht[0] × f[-p, t] × Phi[h, p, -p, p]
```

Out[ ]= f[-p, t] × Phi[h, p, -p, p] × tht[0]

```
In[ ]:= fr0 = t ^ (m0 + 1) wh[kap, nu / 2, 2 Pi Abs[ell] t ^ 2]
        fr1 = t ^ (m0 + 1) wh[kap - 1, nu / 2, 2 Pi Abs[ell] t ^ 2]
```

Out[ ]= $t^{1+m0}\ \text{wh}\left[\text{kap},\ \frac{nu}{2},\ 2\,\pi\,t^2\ \text{Abs}[\text{ell}]\right]$

Out[ ]= $t^{1+m0}\ \text{wh}\left[-1+\text{kap},\ \frac{nu}{2},\ 2\,\pi\,t^2\ \text{Abs}[\text{ell}]\right]$

```
In[ ]:= sh[-3, 1, F, subnab] /. eps → -1
FP1 = Coefficient[%, Phi[-3 + h, 1 + p, -1 - p, 1 + p]] /. f[-p, t] → fr0 /.
    f^(0,ee_)[-p, t] :→ D[fr0, {t, ee}] /. whsub // Simplify
```

Out[ ]= $-\frac{1}{4}\left(h-p+4\,\mathrm{ell}\,\pi\,t^{2}\right) f[-p,t]\times \mathrm{Phi}[-3+h,1+p,-1-p,1+p]\times \mathrm{tht}[0]+\frac{1}{2}\,t\,\mathrm{Phi}[-3+h,1+p,-1-p,1+p]\times \mathrm{tht}[0]\,f^{(0,1)}[-p,t]$

Out[ ]= $-\frac{1}{4}\,t^{1+m0}\,\mathrm{tht}[0]\left(\left(-2+h-2\,m0-p+4\,\mathrm{ell}\,\pi\,t^{2}\right)\mathrm{wh}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\,\pi\,t^{2}\,\mathrm{Abs}[\mathrm{ell}]\right]-8\,\pi\,t^{2}\,\mathrm{Abs}[\mathrm{ell}]\,\mathrm{wh}^{(0,0,1)}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},2\,\pi\,t^{2}\,\mathrm{Abs}[\mathrm{ell}]\right]\right)$

```
In[ ]:= eqn = FP1 - factor tht[0] fr1 //.
    {Abs[ell] → -ell, h → 2 j - 3 p, p → -kap + (j - 1) / 2, m0 → 0} // Simplify
```

Out[ ]= $-t\,\mathrm{tht}[0]\left(\mathrm{factor}\,\mathrm{wh}\left[-1+\mathrm{kap},\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi\,t^{2}\right]+\left(\mathrm{kap}+\mathrm{ell}\,\pi\,t^{2}\right)\mathrm{wh}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi\,t^{2}\right]+2\,\mathrm{ell}\,\pi\,t^{2}\,\mathrm{wh}^{(0,0,1)}\left[\mathrm{kap},\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi\,t^{2}\right]\right)$

The three cases :

```
In[ ]:= eqn /. wh → WhittakerW /. Whdn[kap + 1] /. kap → -p + (j - 1) / 2 // Simplify
Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\frac{1}{4}\times\left(-4+4\,\mathrm{factor}-j^{2}+\mathrm{nu}^{2}-8\,p-4\,p^{2}+4\,j\,(1+p)\right)t\,\mathrm{tht}[0]\,\mathrm{WhittakerW}\left[\frac{1}{2}\times(-3+j-2\,p),\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi\,t^{2}\right]$

Out[ ]= $\left\{\mathrm{factor}\to\frac{1}{4}\times(-2+j-\mathrm{nu}-2\,p)\times(-2+j+\mathrm{nu}-2\,p)\right\}$

```
In[ ]:= eqn /. wh → WhittakerV //. Whrel /. Whdn[kap + 1] /. kap → -p + (j - 1) / 2 // Simplify
Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\left((1+\mathrm{factor})\,t\,\mathrm{tht}[0]\times\mathrm{WhittakerV}\left[\frac{1}{2}\times(-3+j-2\,p),\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi\,t^{2}\right]\right)$

Out[ ]= $\{\mathrm{factor}\to-1\}$

```
In[ ]:= eqn /. wh → WhittakerM /. Whdn[kap + 1] /. kap → -p + (j - 1) / 2 // Simplify
Solve[% == 0, factor][[1]] // Factor
```

Out[ ]= $-\frac{1}{2}\times(-2+2\,\mathrm{factor}+j-\mathrm{nu}-2\,p)\,t\,\mathrm{tht}[0]\,\mathrm{WhittakerM}\left[\frac{1}{2}\times(-3+j-2\,p),\frac{\mathrm{nu}}{2},-2\,\mathrm{ell}\,\pi\,t^{2}\right]$

Out[ ]= $\left\{\mathrm{factor}\to\frac{1}{2}\times(2-j+\mathrm{nu}+2\,p)\right\}$

Box 8 on the right in Table 4.8