22a. Case $0 < m_0(j_1) < p$, and $\varepsilon = 1$

Now r_0 is active in restricting the components. We consider the two lowest components of F.

In[ ]:=
```
Clear[f, fp]
F = tht[0] × f[r0, t] × Phi[h, p, r0, p] + tht[1] × f[r0 + 2, t] × Phi[h, p, r0 + 2, p]
```

Out[ ]= f[r0, t] × Phi[h, p, r0, p] × tht[0] + f[2 + r0, t] × Phi[h, p, 2 + r0, p] × tht[1]

We use that one shift operator gives zero :

In[ ]:=
```
sh[3, -1, F, subnab] /. eps → 1 // Simplify (* should be zero *)
Coefficient[%, Phi[3 + h, -1 + p, 1 + r0, -1 + p]]
fr0p2 = f[2 + r0, t] /. Solve[% == 0, f[2 + r0, t]][[1]] // Simplify
```

Out[ ]=
$$-\frac{1}{4\times(1+p)} p\Big((4-h+2p+r0-4\,\text{ell}\,\pi t^2)\, f[r0,t]\times \text{Phi}[3+h,-1+p,1+r0,-1+p]\times \text{tht}[0]+$$
$$4\,i\sqrt{2\pi}\, t\sqrt{\text{Abs[ell]}}\, f[2+r0,t]\times \text{Phi}[3+h,-1+p,1+r0,-1+p]\times \text{tht}[0]+$$
$$6\, f[2+r0,t]\times \text{Phi}[3+h,-1+p,3+r0,-1+p]\times \text{tht}[1]-$$
$$h\, f[2+r0,t]\times \text{Phi}[3+h,-1+p,3+r0,-1+p]\times \text{tht}[1]+$$
$$2\,p\, f[2+r0,t]\times \text{Phi}[3+h,-1+p,3+r0,-1+p]\times \text{tht}[1]+$$
$$r0\, f[2+r0,t]\times \text{Phi}[3+h,-1+p,3+r0,-1+p]\times \text{tht}[1]-$$
$$4\,\text{ell}\,\pi t^2\, f[2+r0,t]\times \text{Phi}[3+h,-1+p,3+r0,-1+p]\times \text{tht}[1]-$$
$$2\,t\,\text{Phi}[3+h,-1+p,1+r0,-1+p]\times \text{tht}[0]\, f^{(0,1)}[r0,t]-$$
$$2\,t\,\text{Phi}[3+h,-1+p,3+r0,-1+p]\times \text{tht}[1]\, f^{(0,1)}[2+r0,t]\Big)$$

Out[ ]=
$$-\frac{1}{4\times(1+p)} p\Big((4-h+2p+r0-4\,\text{ell}\,\pi t^2)\, f[r0,t]\times \text{tht}[0]+$$
$$4\,i\sqrt{2\pi}\, t\sqrt{\text{Abs[ell]}}\, f[2+r0,t]\times \text{tht}[0]-2\,t\,\text{tht}[0]\, f^{(0,1)}[r0,t]\Big)$$

Out[ ]=
$$-\Big(\big(i\,((-4+h-2p-r0+4\,\text{ell}\,\pi t^2)\, f[r0,t]+2\,t\, f^{(0,1)}[r0,t])\big)\Big/\Big(4\sqrt{2\pi}\, t\sqrt{\text{Abs[ell]}}\Big)\Big)$$

Out[ ]=
$$-\Big(\big(i\,((-4+h-2p-r0+4\,\text{ell}\,\pi t^2)\, f[r0,t]+2\,t\, f^{(0,1)}[r0,t])\big)\Big/\Big(4\sqrt{2\pi}\, t\sqrt{\text{Abs[ell]}}\Big)\Big)$$

We get F^{+} by the other downward shift operator.

In[ ]:=
```
Fp = sh[-3, -1, F, subnab] /. eps → 1 // Simplify
fp[r0 - 1, t] = Coefficient[Fp, Phi[-3 + h, -1 + p, -1 + r0, -1 + p]] / tht[0] // Simplify
```

Out[ ]=

$$-\frac{1}{4\times(1+p)} p\Big(f[r0,t]\big((4+h+2p-r0+4\,ell\,\pi t^2)\,Phi[-3+h,-1+p,-1+r0,-1+p]\times tht[0]-4\,i\,\sqrt{2\pi}\,t\sqrt{Abs[ell]}\,Phi[-3+h,-1+p,1+r0,-1+p]\times tht[1]\big)+f[2+r0,t]\big((2+h+2p-r0+4\,ell\,\pi t^2)\,Phi[-3+h,-1+p,1+r0,-1+p]\times tht[1]-8\,i\sqrt{\pi}\,t\sqrt{Abs[ell]}\,Phi[-3+h,-1+p,3+r0,-1+p]\times tht[2]\big)-2t\big(Phi[-3+h,-1+p,-1+r0,-1+p]\times tht[0]\,f^{(0,1)}[r0,t]+Phi[-3+h,-1+p,1+r0,-1+p]\times tht[1]\,f^{(0,1)}[2+r0,t]\big)\Big)$$

Out[ ]=

$$\frac{-p\,(4+h+2p-r0+4\,ell\,\pi t^2)\,f[r0,t]+2\,p\,t\,f^{(0,1)}[r0,t]}{4\times(1+p)}$$

We know the lowest order term of $b_\upsilon{}^{+}$.

Hence we have with a non-zero factor **cnz**:

In[ ]:=
```
Clear[cnz]
eq = (- (4 + h + 2 p - r0 + 4 ell π t^2) f[r0, t] + 2 t f^(0,1)[r0, t]) ==
      cnz t^(0 + 1) WhittakerV[-p + 1 - (j1 + 1) / 2, nu1, 2 Pi Abs[ell] t^2] /.
    Abs[ell] → ell /. hpsub /. Abs[h + p] → h + p // Simplify
(* Use that V is even in the second parameter. *)
```

Out[ ]=

$$2\,t\,f^{(0,1)}[r0,t]==(4+h+2p-r0+4\,ell\,\pi t^2)\,f[r0,t]+cnz\,t\,WhittakerV\Big[\frac{1}{4}\times(2-h-p),\frac{h+p}{2},2\,ell\,\pi t^2\Big]$$

We solve for the derivative of f_{r_0}

In[ ]:=
```
fr0d = f^(0,1)[r0, t] /. Solve[eq, f^(0,1)[r0, t]][[1]] // Simplify
```

Out[ ]=

$$\frac{1}{2t}\Big((4+h+2p-r0+4\,ell\,\pi t^2)\,f[r0,t]+cnz\,t\,WhittakerV\Big[\frac{1}{4}\times(2-h-p),\frac{h+p}{2},2\,ell\,\pi t^2\Big]\Big)$$

This can be inserted in the eigenfunction equations for $r = r_0$, together with the expressions for f_{r_0+2}

```
In[ ]:= ei = efeqn[h, p, r0, f, ell, 0, 1] /. j → j1 /. nu → nu1 /. f[r0 + 2, t] → fr0p2 /.
          f^(0,1)[2 + r0, t] → D[fr0p2, t] /. f^(0,2)[r0, t] → D[fr0d, t] /.
        f^(0,1)[r0, t] → fr0d //. Whrel /. hpsub /. Abs[h + p]^2 → (h + p)^2 // Simplify
```

Out[ ]= $$\Big\{\frac{1}{16}\,\text{cnz}\,t\Big(4\times(-4+2h+p+r0+8\,\text{ell}\,\pi\,t^2)\,\text{WhittakerV}\Big[\frac{1}{4}\times(2-h-p),\,\frac{h+p}{2},\,2\,\text{ell}\,\pi\,t^2\Big]-(-16+3h^2+8p+3p^2+h(8+6p))\,\text{WhittakerV}\Big[\frac{1}{4}\times(6-h-p),\,\frac{h+p}{2},\,2\,\text{ell}\,\pi\,t^2\Big]\Big),$$
$$\frac{3}{16}\,\text{cnz}\,(p-r0)\,t\Big(-4\times(-4+2h+p+r0+8\,\text{ell}\,\pi\,t^2)\,\text{WhittakerV}\Big[\frac{1}{4}\times(2-h-p),\,\frac{h+p}{2},\,2\,\text{ell}\,\pi\,t^2\Big]+(-16+3h^2+8p+3p^2+h(8+6p))\,\text{WhittakerV}\Big[\frac{1}{4}\times(6-h-p),\,\frac{h+p}{2},\,2\,\text{ell}\,\pi\,t^2\Big]\Big)\Big\}$$

These expressions should be zero. We insert the asymptotic behavior of the V-Whittaker function.

```
In[ ]:= Clear[tau]
        eia = ei /. t → Sqrt[tau] / Sqrt[2 Pi Abs[ell]] /. Abs[ell] → ell /.
           WhittakerV[kap_, s_, tau_] :→ -E^(-Pi I kap) tau^(-kap) E^(tau / 2) // Simplify
```

Out[ ]= $$\Big\{\frac{1}{16\sqrt{\text{ell}}\,\sqrt{2\pi}}\,\text{cnz}\,e^{\text{tau}/2}\,\text{tau}^{\frac{1}{4}\times(-4+h+p)}\Big(e^{\frac{1}{4}i(-6+h+p)\pi}(-16+3h^2+8p+3p^2+h(8+6p))-4\,e^{\frac{1}{4}i(-2+h+p)\pi}\,\text{tau}\,(-4+2h+p+r0+4\,\text{tau})\Big),$$
$$\frac{1}{16\sqrt{\text{ell}}\,\sqrt{2\pi}}\,3\,\text{cnz}\,e^{\text{tau}/2}\,(p-r0)\,\text{tau}^{\frac{1}{4}\times(-4+h+p)}\Big(-e^{\frac{1}{4}i(-6+h+p)\pi}(-16+3h^2+8p+3p^2+h(8+6p))+4\,e^{\frac{1}{4}i(-2+h+p)\pi}\,\text{tau}\,(-4+2h+p+r0+4\,\text{tau})\Big)\Big\}$$

```
In[ ]:= eia E^(-tau / 2) tau^(-1 - (h + p) / 4) // Simplify
        Limit[%, tau → Infinity]
```

Out[ ]= $$\Big\{\Big(i\,\text{cnz}\,e^{\frac{1}{4}i(h+p)\pi}\big(3h^2+3p^2+4p(2+\text{tau})+h(8+6p+8\,\text{tau})+4\times(-4+(-4+r0)\,\text{tau}+4\,\text{tau}^2)\big)\Big)\Big/\Big(16\sqrt{\text{ell}}\,\sqrt{2\pi}\,\text{tau}^2\Big),\ -\Big(\Big(3\,i\,\text{cnz}\,e^{\frac{1}{4}i(h+p)\pi}(p-r0)\big(3h^2+3p^2+4p(2+\text{tau})+h(8+6p+8\,\text{tau})+4\times(-4+(-4+r0)\,\text{tau}+4\,\text{tau}^2)\big)\Big)\Big/\Big(16\sqrt{\text{ell}}\,\sqrt{2\pi}\,\text{tau}^2\Big)\Big)\Big\}$$

Out[ ]= $$\Big\{\frac{i\,\text{cnz}\,e^{\frac{1}{4}i(h+p)\pi}}{\sqrt{\text{ell}}\,\sqrt{2\pi}},\ \frac{-48\,i\,\text{cnz}\,e^{\frac{1}{4}i(h+p)\pi}\,p+48\,i\,\text{cnz}\,e^{\frac{1}{4}i(h+p)\pi}\,r0}{16\sqrt{\text{ell}}\,\sqrt{2\pi}}\Big\}$$

So **cnz** has to vanish.