## 22e. Case $0 \le m_0(j_2) < m_0(j_1)$ and $\varepsilon = 1$

In[ ]:=
```
m0[j1] /. mrsub /. eps → 1
m0[j2] /. mrsub /. eps → 1
m0[j1] - m0[j2] /. mrsub /. eps → 1 // Simplify
```

Out[ ]= $\frac{p - r0}{2}$

Out[ ]= $-p - r0$

Out[ ]= $\frac{1}{2} \times (3\,p + r0)$

This shows that $r_0 \le -p$ .

The lowest component of F is $f_{-p}$ and the highest component is $f_p$. These orders are different, since p≥1.

We denote $F^+$ by **Fp**, and $F^-$ by **Fm**.

The components of $F^\pm$ depend on two components of $F$.

In[ ]:=
```
Clear[F, Fp, Fm, f, fph, fml]
F = tht[m[h, p]] × f[p, t] × Phi[h, p, p, p] + tht[m[h, p] - 1] × f[p - 2, t] × Phi[h, p, p - 2, p]
sh[3, -1, F, subnab] /. eps → 1 /. j → j2 // Simplify
fph = (* highest component of Fp *)
 Coefficient[%, Phi[3 + h, -1 + p, -1 + p, -1 + p]] / tht[m[h, p] - 1] // Simplify
```

Out[ ]= f[-2 + p, t] × Phi[h, p, -2 + p, p] × tht[-1 + m[h, p]] + f[p, t] × Phi[h, p, p, p] × tht[m[h, p]]

Out[ ]= $\frac{1}{4 \times (1+p)} p \Big(-4\,i\,\sqrt{2\,\pi}\;t\,\sqrt{\text{Abs[ell]}}$
$\big(\text{f[-2 + p, t]}\,\sqrt{-1 + \text{m[h, p]}}\;\text{Phi[3 + h, -1 + p, -3 + p, -1 + p]} \times \text{tht[-2 + m[h, p]]} +$
$\text{f[p, t]}\,\sqrt{\text{m[h, p]}}\;\text{Phi[3 + h, -1 + p, -1 + p, -1 + p]} \times \text{tht[-1 + m[h, p]]}\big) +$
$\text{Phi[3 + h, -1 + p, -1 + p, -1 + p]} \times \text{tht[-1 + m[h, p]]}$
$\big((-2 + h - 3\,p + 4\,\text{ell}\,\pi\,t^2)\,\text{f[-2 + p, t]} + 2\,t\,f^{(0,1)}\text{[-2 + p, t]}\big)\Big)$

Out[ ]= $\frac{1}{4 \times (1+p)}$
$p\,\Big((-2 + h - 3\,p + 4\,\text{ell}\,\pi\,t^2)\,\text{f[-2 + p, t]} + 2\,t\,\big(-2\,i\,\sqrt{2\,\pi}\;\sqrt{\text{Abs[ell]}}\;\text{f[p, t]}\,\sqrt{\text{m[h, p]}} + f^{(0,1)}\text{[-2 + p, t]}\big)\Big)$

In[ ]:=
```
F = tht[m[h, -p]] × f[-p, t] × Phi[h, p, -p, p] + tht[m[h, -p] + 1] × f[2 - p, t] × Phi[h, p, 2 - p, p]
sh[-3, -1, F, subnab] /. eps → 1 /. j → j1 // Simplify
fml = (* lowest component of Fm *)
 Coefficient[%, Phi[-3 + h, -1 + p, 1 - p, -1 + p]] / tht[m[h, -p] + 1] // Simplify
```

Out[ ]= f[-p, t] × Phi[h, p, -p, p] × tht[m[h, -p]] + f[2 - p, t] × Phi[h, p, 2 - p, p] × tht[1 + m[h, -p]]

Out[ ]=
$$\frac{1}{4\times(1+p)} p\Big(4 i\sqrt{2\pi}\ t\sqrt{\text{Abs[ell]}}\Big(f[-p,t]\sqrt{1+m[h,-p]}\ \text{Phi}[-3+h,-1+p,1-p,-1+p]\times\text{tht}[1+m[h,-p]]+ f[2-p,t]\sqrt{2+m[h,-p]}\ \text{Phi}[-3+h,-1+p,3-p,-1+p]\times\text{tht}[2+m[h,-p]]\Big)- \text{Phi}[-3+h,-1+p,1-p,-1+p]\times\text{tht}[1+m[h,-p]]\left(\left(2+h+3p+4\,\text{ell}\,\pi t^2\right)f[2-p,t]-2t\,f^{(0,1)}[2-p,t]\right)\Big)$$

Out[ ]=
$$\frac{1}{4\times(1+p)} p\left(-\left(\left(2+h+3p+4\,\text{ell}\,\pi t^2\right)f[2-p,t]\right)+ 2t\left(2 i\sqrt{2\pi}\sqrt{\text{Abs[ell]}}\ f[-p,t]\sqrt{1+m[h,-p]}+f^{(0,1)}[2-p,t]\right)\right)$$

We assume that the derivatives $F^{+}$and $F^{-}$ are a linear combination of basis functions, with determining components as indicated in Table 4.9.
The K-type of $F^{+}$corresponds to a point on the left boundary of the sector Sect$(j_2)$.
For $F^{+}$ we have to use the lower part of Table 4.9, applying it with $x^{0,p-1}$.

In[ ]:=
```
Clear[cop, cup]
eqp = (fph == t ^ (p - 1 + 1) (cop WhittakerW [-m0[j2] - (j2 + 1) / 2, nu2 / 2, 2 Pi Abs[ell] t ^ 2] +
        cup WhittakerV [-m0[j2] - (j2 + 1) / 2, nu2 / 2, 2 Pi Abs[ell] t ^ 2])) /.
    m[h, r_] :→ m0[j2] + (1 / 6) × (3 r + 2 j2 - h) /. hpsub /. eps → 1 // Simplify
```

Out[ ]=
$$\frac{1}{4\times(1+p)} p\left(\left(-2+h-3p+4\,\text{ell}\,\pi t^2\right)f[-2+p,t]+ 2t\left(-2 i\sqrt{2\pi}\sqrt{\text{Abs[ell]}}\ f[p,t]\sqrt{p+\text{m0}\left[\tfrac{1}{2}(h+3p)\right]}+f^{(0,1)}[-2+p,t]\right)\right)==$$
$$t^{p}\left(\text{cup WhittakerV}\left[\tfrac{1}{4}\times\left(-2-h-3p-4\,\text{m0}\left[\tfrac{1}{2}(h+3p)\right]\right),\ \tfrac{1}{4}\text{Abs}[h-p],\ 2\pi t^2\,\text{Abs[ell]}\right]+ \text{cop WhittakerW}\left[\tfrac{1}{4}\times\left(-2-h-3p-4\,\text{m0}\left[\tfrac{1}{2}(h+3p)\right]\right),\ \tfrac{1}{4}\text{Abs}[h-p],\ 2\pi t^2\,\text{Abs[ell]}\right]\right)$$

For $F^{-}$we deal with a K-type on the right boundary of the sector Sect$(j_1)$. Since $m_0(j_1)\geq p$ we get

In[ ]:=
```
Clear[com, cum]
eqm = (fml == t^(p - 1 + 1) (com WhittakerW[-m0[j1] - (j1 + 1) / 2, nu1 / 2, 2 Pi Abs[ell] t^2] +
          cum WhittakerV[-m0[j1] - (j1 + 1) / 2, nu1 / 2, 2 Pi Abs[ell] t^2])) /.
    m[h, r_] :> m0[j1] + (1 / 6) × (3 r + 2 j1 - h) //. hpsub /. eps → 1 // Simplify
```

Out[ ]=
$$\frac{1}{4\times(1+p)} p \left(-\left(\left(2+h+3p+4\,\text{ell}\,\pi t^2\right) f[2-p,t]\right) + 2t\left(2 i \sqrt{2\pi}\sqrt{\text{Abs[ell]}}\, f[-p,t]\sqrt{1-p+\text{m0}\left[\frac{1}{2}(h-3p)\right]} + f^{(0,1)}[2-p,t]\right)\right) == t^p\left(\text{cum WhittakerV}\left[\frac{1}{4}\times\left(-2-h+3p-4\,\text{m0}\left[\frac{1}{2}(h-3p)\right]\right), \frac{1}{4}\text{Abs}[h+p], 2\pi t^2\,\text{Abs[ell]}\right] + \text{com WhittakerW}\left[\frac{1}{4}\times\left(-2-h+3p-4\,\text{m0}\left[\frac{1}{2}(h-3p)\right]\right), \frac{1}{4}\text{Abs}[h+p], 2\pi t^2\,\text{Abs[ell]}\right]\right)$$

The quantities **cop**, **cup**, **com** and **cum** are unknown coefficients.
Table 4.5 expresses the parameter $\kappa_0$ in terms of $m_0(j_1)$ and $m_0(j_2)$.
We prefer to leave these quantities $m_0(j_1) = m_0((h-3\,p)/2)$ and $m_0(j_2) = m_0((h+3\,p)/2)$ in the computations.

The resulting equations **eqm** and **eqp** involve the components $f_{\pm p}$ , $f_{\pm(p-2)}$ and their derivatives.
We solve the equations for $f_{\pm(p-2)}$'.

In[ ]:=
```
solp = Solve[eqp, f^(0,1)[-2 + p, t]][[1]] // Simplify
solm = Solve[eqm, f^(0,1)[2 - p, t]][[1]] // Simplify
```

Out[ ]=
$$\left\{f^{(0,1)}[-2+p,t] \to -\frac{\left(-2+h-3p+4\,\text{ell}\,\pi t^2\right) f[-2+p,t]}{2t} + 2 i\sqrt{2\pi}\sqrt{\text{Abs[ell]}}\, f[p,t]\sqrt{p+\text{m0}\left[\frac{1}{2}(h+3p)\right]} + \frac{1}{p}2\times(1+p)\,t^{-1+p}\left(\text{cup WhittakerV}\left[\frac{1}{4}\times\left(-2-h-3p-4\,\text{m0}\left[\frac{1}{2}(h+3p)\right]\right), \frac{1}{4}\text{Abs}[h-p], 2\pi t^2\,\text{Abs[ell]}\right] + \text{cop WhittakerW}\left[\frac{1}{4}\times\left(-2-h-3p-4\,\text{m0}\left[\frac{1}{2}(h+3p)\right]\right), \frac{1}{4}\text{Abs}[h-p], 2\pi t^2\,\text{Abs[ell]}\right]\right)\right\}$$

Out[ ]=
$$\left\{f^{(0,1)}[2-p,t] \to \frac{\left(2+h+3p+4\,\text{ell}\,\pi t^2\right) f[2-p,t]}{2t} - 2 i\sqrt{2\pi}\sqrt{\text{Abs[ell]}}\, f[-p,t]\sqrt{1-p+\text{m0}\left[\frac{1}{2}(h-3p)\right]} + \frac{1}{p}2\times(1+p)\,t^{-1+p}\left(\text{cum WhittakerV}\left[\frac{1}{4}\times\left(-2-h+3p-4\,\text{m0}\left[\frac{1}{2}(h-3p)\right]\right), \frac{1}{4}\text{Abs}[h+p], 2\pi t^2\,\text{Abs[ell]}\right] + \text{com WhittakerW}\left[\frac{1}{4}\times\left(-2-h+3p-4\,\text{m0}\left[\frac{1}{2}(h-3p)\right]\right), \frac{1}{4}\text{Abs}[h+p], 2\pi t^2\,\text{Abs[ell]}\right]\right)\right\}$$

We take the eigenfunction equations for the components of F, of order p for the + case, and of order -p for the - case.

In[ ]:=
```
eip = efeqn[h, p, p, f, ell, m[p], 1] /. solp /. nu → nu2 /. j → j2 /. {eps → 1, Abs[ell] → ell} /.
    m[x_] :→ m0[j2] + (1 / 6) × (3 x + 2 j2 - h) //. hpsub /. Abs[h - p]^2 → (h - p)^2 // Simplify
eim = efeqn[h, p, -p, f, ell, m[-p], 1] /. solm /. nu → nu1 /. j → j1 /.
    {eps → 1, Abs[ell] → ell} /. m[x_] :→ m0[j1] + (1 / 6) × (3 x + 2 j1 - h) //.
   hpsub /. Abs[h + p]^2 → (h + p)^2 // Simplify
```

Out[ ]=

$$\Big\{-\frac{1}{4}\, f[p, t]\Big(-16 + h^2 + 2\, h\, p + p^2 + 16\, ell\, \pi\, t^2 + 8\, ell\, h\, \pi\, t^2 + 40\, ell\, p\, \pi\, t^2 + 16\, ell^2\, \pi^2\, t^4 + 32\, ell\, \pi\, t^2\, m0\Big[\frac{1}{2}\,(h + 3\, p)\Big]\Big) + t\Big(4\, i\, \sqrt{ell}\, p\, \sqrt{2\,\pi}\, f[-2 + p, t]\, \sqrt{p + m0\Big[\frac{1}{2}\,(h + 3\, p)\Big]} - 3\, f^{(0,1)}[p, t] + t\, f^{(0,2)}[p, t]\Big),$$
$$-24\, i\, \sqrt{ell}\,(1 + p)\,\sqrt{2\,\pi}\, t^{1+p}\, \sqrt{p + m0\Big[\frac{1}{2}\,(h + 3\, p)\Big]}\,\Big(cup\, WhittakerV\Big[\frac{1}{4} \times \Big(-2 - h - 3\, p - 4\, m0\Big[\frac{1}{2}\,(h + 3\, p)\Big]\Big), \frac{1}{4}\, Abs[h - p], 2\, ell\, \pi\, t^2\Big] + cop\, WhittakerW\Big[\frac{1}{4} \times \Big(-2 - h - 3\, p - 4\, m0\Big[\frac{1}{2}\,(h + 3\, p)\Big]\Big), \frac{1}{4}\, Abs[h - p], 2\, ell\, \pi\, t^2\Big]\Big)\Big\}$$

Out[ ]=

$$\Big\{-\frac{1}{4}\, f[-p, t]\Big(-16 + h^2 - 2\, h\, p + p^2 + 16\, ell\, \pi\, t^2 + 8\, ell\, h\, \pi\, t^2 - 40\, ell\, p\, \pi\, t^2 + 16\, ell^2\, \pi^2\, t^4 + 32\, ell\, \pi\, t^2\, m0\Big[\frac{1}{2}\,(h - 3\, p)\Big]\Big) + t\Big(-4\, i\, \sqrt{ell}\, p\, \sqrt{2\,\pi}\, f[2 - p, t]\, \sqrt{1 - p + m0\Big[\frac{1}{2}\,(h - 3\, p)\Big]} - 3\, f^{(0,1)}[-p, t] + t\, f^{(0,2)}[-p, t]\Big),$$
$$-24\, i\, \sqrt{ell}\,(1 + p)\,\sqrt{2\,\pi}\, t^{1+p}\, \sqrt{1 - p + m0\Big[\frac{1}{2}\,(h - 3\, p)\Big]}\,\Big(cum\, WhittakerV\Big[\frac{1}{4} \times \Big(-2 - h + 3\, p - 4\, m0\Big[\frac{1}{2}\,(h - 3\, p)\Big]\Big), \frac{1}{4}\, Abs[h + p], 2\, ell\, \pi\, t^2\Big] + com\, WhittakerW\Big[\frac{1}{4} \times \Big(-2 - h + 3\, p - 4\, m0\Big[\frac{1}{2}\,(h - 3\, p)\Big]\Big), \frac{1}{4}\, Abs[h + p], 2\, ell\, \pi\, t^2\Big]\Big)\Big\}$$

The following quantities should vanish.

In[ ]:=
```
ei = {eip⟦2⟧, eim⟦2⟧} /. tht[mm_] :→ 1
```

Out[ ]=

$$\Big\{-24\, i \sqrt{\mathtt{ell}}\;(1+p)\,\sqrt{2\pi}\; t^{1+p}\,\sqrt{p+\mathtt{m0}\Big[\frac{1}{2}\,(h+3\,p)\Big]}$$
$$\Big(\mathtt{cup}\;\mathtt{WhittakerV}\Big[\frac{1}{4}\times\Big(-2-h-3\,p-4\,\mathtt{m0}\Big[\frac{1}{2}\,(h+3\,p)\Big]\Big),\ \frac{1}{4}\,\mathtt{Abs}[h-p],\ 2\,\mathtt{ell}\,\pi\, t^2\Big]+$$
$$\mathtt{cop}\;\mathtt{WhittakerW}\Big[\frac{1}{4}\times\Big(-2-h-3\,p-4\,\mathtt{m0}\Big[\frac{1}{2}\,(h+3\,p)\Big]\Big),\ \frac{1}{4}\,\mathtt{Abs}[h-p],\ 2\,\mathtt{ell}\,\pi\, t^2\Big]\Big),$$
$$-24\, i \sqrt{\mathtt{ell}}\;(1+p)\,\sqrt{2\pi}\; t^{1+p}\,\sqrt{1-p+\mathtt{m0}\Big[\frac{1}{2}\,(h-3\,p)\Big]}$$
$$\Big(\mathtt{cum}\;\mathtt{WhittakerV}\Big[\frac{1}{4}\times\Big(-2-h+3\,p-4\,\mathtt{m0}\Big[\frac{1}{2}\,(h-3\,p)\Big]\Big),\ \frac{1}{4}\,\mathtt{Abs}[h+p],\ 2\,\mathtt{ell}\,\pi\, t^2\Big]+$$
$$\mathtt{com}\;\mathtt{WhittakerW}\Big[\frac{1}{4}\times\Big(-2-h+3\,p-4\,\mathtt{m0}\Big[\frac{1}{2}\,(h-3\,p)\Big]\Big),\ \frac{1}{4}\,\mathtt{Abs}[h+p],\ 2\,\mathtt{ell}\,\pi\, t^2\Big]\Big)\Big\}$$

Use the asymptotic behavior of the Whittaker functions. First look at the exponentially growing contributions.

In[ ]:=
```
ei /. WhittakerW[kp_, s_, tau_] :→ 0 /. WhittakerV[kp_, s_, tau_] :→ tau^(-kp) Exp[tau/2] /.
   (ell t^2)^ee_ :→ ell^ee t^(2 ee) // Simplify
Series[Simplify[E^(-Pi ell t^2) %], {t, Infinity, 0}] // Simplify
```

Out[ ]=

$$\Big\{-3\, i\, 2^{\frac{1}{4}\times(16+h+3\,p)+\mathtt{m0}\left[\frac{1}{2}\,(h+3\,p)\right]}\ \mathtt{cup}\ e^{\mathtt{ell}\,\pi\, t^2}\ \mathtt{ell}^{\frac{1}{4}\times\left(4+h+3\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h+3\,p)\right]\right)}$$
$$(1+p)\,\pi^{\frac{1}{4}\times\left(4+h+3\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h+3\,p)\right]\right)}\ t^{\frac{1}{2}\times\left(4+h+5\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h+3\,p)\right]\right)}\ \sqrt{p+\mathtt{m0}\Big[\frac{1}{2}\,(h+3\,p)\Big]}\ ,$$
$$-3\, i\, 2^{\frac{1}{4}\times(16+h-3\,p)+\mathtt{m0}\left[\frac{1}{2}\,(h-3\,p)\right]}\ \mathtt{cum}\ e^{\mathtt{ell}\,\pi\, t^2}\ \mathtt{ell}^{\frac{1}{4}\times\left(4+h-3\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h-3\,p)\right]\right)}\,(1+p)$$
$$\pi^{\frac{1}{4}\times\left(4+h-3\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h-3\,p)\right]\right)}\ t^{\frac{1}{2}\times\left(4+h-p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h-3\,p)\right]\right)}\ \sqrt{1-p+\mathtt{m0}\Big[\frac{1}{2}\,(h-3\,p)\Big]}\ \Big\}$$

Out[ ]=

$$\Big\{-3\, i\, 2^{\frac{1}{4}\times(16+h+3\,p)+\mathtt{m0}\left[\frac{1}{2}\,(h+3\,p)\right]}\ \mathtt{cup}\ \mathtt{ell}^{\frac{1}{4}\times\left(4+h+3\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h+3\,p)\right]\right)}\,(1+p)\,\pi^{\frac{1}{4}\times\left(4+h+3\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h+3\,p)\right]\right)}$$
$$t^{\frac{1}{2}\times\left(4+h+5\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h+3\,p)\right]\right)}\ \sqrt{p+\mathtt{m0}\Big[\frac{1}{2}\,(h+3\,p)\Big]}\ ,\ -3\, i\, 2^{\frac{1}{4}\times(16+h-3\,p)+\mathtt{m0}\left[\frac{1}{2}\,(h-3\,p)\right]}\ \mathtt{cum}$$
$$\mathtt{ell}^{\frac{1}{4}\times\left(4+h-3\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h-3\,p)\right]\right)}\,(1+p)\,\pi^{\frac{1}{4}\times\left(4+h-3\,p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h-3\,p)\right]\right)}\ t^{\frac{1}{2}\times\left(4+h-p+4\,\mathtt{m0}\left[\frac{1}{2}\,(h-3\,p)\right]\right)}\ \sqrt{1-p+\mathtt{m0}\Big[\frac{1}{2}\,(h-3\,p)\Big]}\ \Big\}$$

This shows that the coefficients **cum** and **cup** vanish.

Now the contribution of the W-Whittaker functions.

In[ ]:=
```
ei /. {cum → 0, cup → 0} /. WhittakerW[kp_, s_, tau_] :→ tau ^ kp Exp[- tau / 2] /.
  (ell t^2) ^ ee_ :→ ell ^ ee t ^ (2 ee) // Simplify
```

Out[ ]=

$$\left\{-3\, i\, 2^{3-\frac{h}{4}-\frac{3p}{4}-\mathrm{m0}\left[\frac{1}{2}(h+3p)\right]}\,\mathrm{cop}\, e^{-\mathrm{ell}\,\pi\, t^2}\,\mathrm{ell}^{\frac{1}{4}\left(-h-3p-4\,\mathrm{m0}\left[\frac{1}{2}(h+3p)\right]\right)}(1+p)\,\pi^{\frac{1}{4}\left(-h-3p-4\,\mathrm{m0}\left[\frac{1}{2}(h+3p)\right]\right)}\, t^{\frac{1}{2}\left(-h-p-4\,\mathrm{m0}\left[\frac{1}{2}(h+3p)\right]\right)}\sqrt{p+\mathrm{m0}\left[\frac{1}{2}(h+3p)\right]},\ -3\, i\, 2^{3-\frac{h}{4}+\frac{3p}{4}-\mathrm{m0}\left[\frac{1}{2}(h-3p)\right]}\,\mathrm{com}\, e^{-\mathrm{ell}\,\pi\, t^2}\,\mathrm{ell}^{\frac{1}{4}\left(-h+3p-4\,\mathrm{m0}\left[\frac{1}{2}(h-3p)\right]\right)}(1+p)\,\pi^{\frac{1}{4}\left(-h+3p-4\,\mathrm{m0}\left[\frac{1}{2}(h-3p)\right]\right)}\, t^{\frac{1}{2}\left(-h+5p-4\,\mathrm{m0}\left[\frac{1}{2}(h-3p)\right]\right)}\sqrt{1-p+\mathrm{m0}\left[\frac{1}{2}(h-3p)\right]}\right\}$$

This shows that the coefficient **cop** and **cup** vanish as well.

The conclusion is that the determining coefficients of $F^+$ and $F^-$ are zero, which completes the proof in this case.