4b. Lie algebra action on basis polynomials

See §2.2.1 and §2.2.2

We need the action of **WW1** and **WW2** with exponentials **km**[1,Cos[t],Sin[t]] and **km**[1,Cos[t],I Sin[t]]
These elements are in SU(2) and we can work with the generating function in (2.24).

The generating function

```
In[ ]:= Clear[a, b, c, d, x, p, q, r]
g = {{a, b, 0}, {c, d, 0}, {0, 0, 1}};
Ph[a_, b_, c_, d_, p_, q_] := (a x + c)^((p - q)/2) (b x + d)^((p + q)/2)
```

Right differentiate with respect to **WW1** and **WW2**

Compute right translation

```
In[ ]:= u = {{a, b, 0}, {c, d, 0}, {0, 0, 1}}
v = u.km[1, Cos[t], Sin[t]] /. Conjugate[xx_] :> xx
Out[ ]= {{a, b, 0}, {c, d, 0}, {0, 0, 1}}
Out[ ]= {{a Cos[t] - b Sin[t], b Cos[t] + a Sin[t], 0}, {c Cos[t] - d Sin[t], d Cos[t] + c Sin[t], 0}, {0, 0, 1}}
```

Insert result

```
In[ ]:= dPh1 =
 D[Ph[a Cos[t] - b Sin[t], b Cos[t] + a Sin[t], c Cos[t] - d Sin[t], d Cos[t] + c Sin[t], p, q], t] /.
   t -> 0 // Simplify
```

Out[ ]= $$\frac{1}{2} (c + a\,x)^{\frac{1}{2}\times(-2+p-q)} (d + b\,x)^{\frac{p+q}{2}} \left(\frac{(p + q)(c + a\,x)^2}{d + b\,x} - (p - q)(d + b\,x)\right)$$

Similarly for the other Lie algebra element

```
In[ ]:= u = {{a, b, 0}, {c, d, 0}, {0, 0, 1}}
v = u.km[1, Cos[t], I Sin[t]] /. Conjugate[x_] :> x
Out[ ]= {{a, b, 0}, {c, d, 0}, {0, 0, 1}}
Out[ ]= {{a Cos[t] + i b Sin[t], b Cos[t] + i a Sin[t], 0}, {c Cos[t] + i d Sin[t], d Cos[t] + i c Sin[t], 0}, {0, 0, 1}}
```

```
In[ ]:= Clear[dPh2]
dPh2 =
 D[Ph[a Cos[t] + i b Sin[t], b Cos[t] + i a Sin[t], c Cos[t] + i d Sin[t], d Cos[t] + i c Sin[t], p, q],
    t] /. t -> 0 // Simplify
```

Out[ ]= $$\frac{1}{2}\, i\, (c + a\,x)^{\frac{1}{2}\times(-2+p-q)} (d + b\,x)^{\frac{1}{2}\times(-2+p+q)} \left(d^2 (p - q) + c^2 (p + q) + 2\,b\,d\,(p - q)\,x + 2\,a\,c\,(p + q)\,x + \left(b^2 (p - q) + a^2 (p + q)\right) x^2\right)$$

Combination in Z_{12} and Z_{21}.

In[ ]:=
```
Clear[eps]
dPheps = dPh1 + I eps dPh2 // Simplify
```

Out[ ]=
$$\frac{1}{2} (c + a x)^{\frac{1}{2} \times (-2+p-q)} (d + b x)^{\frac{1}{2} \times (-2+p+q)} \left(-d^2 (1 + eps) (p - q) - c^2 (-1 + eps) (p + q) - 2 b d (1 + eps) (p - q) x - 2 a c (-1 + eps) (p + q) x + \left(-b^2 (1 + eps) (p - q) - a^2 (-1 + eps) (p + q)\right) x^2\right)$$

In[ ]:=
```
dPheps /. eps → 1 // Simplify
% == (q - p) Ph[a, b, c, d, p, q + 2] // Simplify
```

Out[ ]= $(-p + q) (c + a x)^{\frac{1}{2} \times (-2+p-q)} (d + b x)^{\frac{1}{2} \times (2+p+q)}$

Out[ ]= True

In[ ]:=
```
dPheps /. eps → -1 // Simplify
% == (q + p) Ph[a, b, c, d, p, q - 2] // Simplify
```

Out[ ]= $(p + q) (c + a x)^{\frac{1}{2} \times (2+p-q)} (d + b x)^{\frac{1}{2} \times (-2+p+q)}$

Out[ ]= True

So the generating function satisfies the desired relations, and hence its coefficients satisfy the relation as well.

In[ ]:=

Left differentiate with respect to **WW1** and **WW2**

Left and right differentiation commute. It suffices to consider q=p

In[ ]:=
```
Clear[Phi]
Phi[r_] = (*This is (Φ^{p/2})_{r/2,p/q} *)Binomial[p, (p - r) / 2] b ^ ((p - r) / 2) d ^ ((p + r) / 2)
```

Out[ ]= $b^{\frac{p-r}{2}} d^{\frac{p+r}{2}} \text{Binomial}\left[p, \frac{p-r}{2}\right]$

In[ ]:=
```
u = {{a, b, 0}, {c, d, 0}, {0, 0, 1}}
v = km[1, Cos[t], Sin[t]].u /. Conjugate[xx_] :→ xx
```

Out[ ]= {{a, b, 0}, {c, d, 0}, {0, 0, 1}}

Out[ ]= {{a Cos[t] + c Sin[t], b Cos[t] + d Sin[t], 0}, {c Cos[t] - a Sin[t], d Cos[t] - b Sin[t], 0}, {0, 0, 1}}

In[ ]:=
```
Phi[r] /. b → b Cos[t] + d Sin[t] /. d → d Cos[t] - b Sin[t]
dPh1 = D[%, t] /. t → 0 // Simplify
```

Out[ ]= $\text{Binomial}\left[p, \frac{p-r}{2}\right] (d \text{Cos}[t] - b \text{Sin}[t])^{\frac{p+r}{2}} (b \text{Cos}[t] + \text{Sin}[t] (d \text{Cos}[t] - b \text{Sin}[t]))^{\frac{p-r}{2}}$

Out[ ]= $-\frac{1}{2} b^{\frac{1}{2} \times (-2+p-r)} d^{\frac{1}{2} \times (-2+p+r)} (d^2 (-p + r) + b^2 (p + r)) \text{Binomial}\left[p, \frac{p-r}{2}\right]$

```
In[ ]:= u = {{a, b, 0}, {c, d, 0}, {0, 0, 1}}
        v = km[1, Cos[t], I Sin[t]].u /. Conjugate[xx_] :> xx
```

Out[ ]= {{a, b, 0}, {c, d, 0}, {0, 0, 1}}

Out[ ]= {{a Cos[t] + ⅈ c Sin[t], b Cos[t] + ⅈ d Sin[t], 0}, {c Cos[t] + ⅈ a Sin[t], d Cos[t] + ⅈ b Sin[t], 0}, {0, 0, 1}}

```
In[ ]:= Phi[r] /. b → b Cos[t] + ⅈ d Sin[t] /. d → d Cos[t] + ⅈ b Sin[t]
        dPh2 = D[%, t] /. t → 0 // Simplify
```

Out[ ]= $\text{Binomial}\left[p, \frac{p-r}{2}\right] (d\,\text{Cos}[t] + i\, b\,\text{Sin}[t])^{\frac{p+r}{2}} (b\,\text{Cos}[t] + i\,\text{Sin}[t]\,(d\,\text{Cos}[t] + i\, b\,\text{Sin}[t]))^{\frac{p-r}{2}}$

Out[ ]= $\frac{1}{2}\, i\, b^{\frac{1}{2}\times(-2+p-r)}\, d^{\frac{1}{2}\times(-2+p+r)} \left(d^2 (p-r) + b^2 (p+r)\right) \text{Binomial}\left[p, \frac{p-r}{2}\right]$

```
In[ ]:= dPh[eps_] = dPh1 + I eps dPh2 // Simplify
```

Out[ ]= $-\frac{1}{2}\, b^{\frac{1}{2}\times(-2+p-r)}\, d^{\frac{1}{2}\times(-2+p+r)} \left(d^2 (-1+\text{eps}) (p-r) + b^2 (1+\text{eps}) (p+r)\right) \text{Binomial}\left[p, \frac{p-r}{2}\right]$

```
In[ ]:= dPh[1]
        % /. r → -p
```

Out[ ]= $-b^{2+\frac{1}{2}\times(-2+p-r)}\, d^{\frac{1}{2}\times(-2+p+r)} (p+r)\, \text{Binomial}\left[p, \frac{p-r}{2}\right]$

Out[ ]= 0

```
In[ ]:= dPh[1] / Phi[r - 2] // FullSimplify
```

Out[ ]= -2 - p + r

```
In[ ]:= dPh[-1] / Phi[r + 2] // FullSimplify
        dPh[-1] /. r → p
```

Out[ ]= 2 + p + r

Out[ ]= 0