7 Shift operators, construction

§3.1

7a. Shift operators in general (**g**,K)-modules

7b. Minimal vectors

See §3.1

Part ii) of lemma 3.4

In[ ]:=
```
Clear[v, h, p]
x1 = sh[-3, -1] ** sh[3, 1] ** v - (p + 1) sh[3, -1] ** sh[-3, 1] ** v /.
      {sh[-3, -1] ** sh[3, 1] ** v → (Z13 - (2 (p + 2)) ^ (-1) Z12 ** Z23) ** Z31 ** v,
       sh[3, -1] ** sh[-3, 1] ** v → (Z32 + (2 (p + 2)) ^ (-1) Z12 ** Z31) ** Z23 ** v} //.
     {CKi ** v → -I h v, WW0 ** v → -I p v, (ff_^(-1) XX_) ** YY_ :→ ff ^ (-1) XX ** YY,
      Z21 ** v → 0} // Expand // Simplify
```

Out[ ]= $\frac{1}{2}\left(-((h-p)(1+p)\, v) - 2\times(1+p)\, Z23 ** Z32 ** v - Z23 ** Z12 ** Z31 ** v\right)$

In[ ]:=
```
x1 /. Z12 ** Z31 ** v → -(2 (p + 1)) Z32 ** v //. {XX_ ** ((p + 1) YY_) :→ (p + 1) XX ** YY} // Simplify
```

Out[ ]= $-\frac{1}{2}(h-p)(1+p)\, v$

In[ ]:=

Casimir element, part iii) of the lemma

In[ ]:=
```
x1 = CasZ ** v - 4 (p + 2) (p + 1) ^ (-1) sh[-3, -1] ** sh[3, 1] ** v
```

Out[ ]= $2\, i\, CKi ** v + 2\, i\, WW0 ** v - \frac{CKi ** CKi ** v}{3} - WW0 ** WW0 ** v - Z12 ** Z21 ** v + 4\, Z13 ** Z31 ** v + 4\, Z23 ** Z32 ** v - \frac{4\times(2+p)\, sh[-3, -1] ** sh[3, 1] ** v}{1+p}$

In[ ]:=
```
x2 = x1 /. {sh[-3, -1] ** sh[3, 1] ** v → (Z13 - (2 (p + 2)) ^ (-1) Z12 ** Z23) ** Z31 ** v} //.
    { CKi ** v → -I h v, WW0 ** v → -I p v, XX_ ** (ff_ v) :→ ff XX ** v,
     Z21 ** v → 0, (ff_^(-1) XX_) ** YY_ :→ ff ^ (-1) XX ** YY} // Simplify
```

Out[ ]= $\frac{1}{3}\times\left(6\, h + h^2 + 3\, p\,(2+p)\right) v + 4\, Z23 ** Z32 ** v + \frac{2\, Z23 ** Z12 ** Z31 ** v}{1+p}$

Use of minimality

In[ ]:=
```
x2 /. Z12 ** Z31 ** v → -(2 (p + 1)) Z32 ** v //. {XX_ ** ((1 + p) YY_) :→ (1 + p) XX ** YY} // Expand // Factor
```

Out[ ]=
$$\frac{1}{3} \times \left(6\,h + h^2 + 6\,p + 3\,p^2\right) v$$

In[ ]:=

Central element of degree 3, part iv) of the lemma

In a similar way

In[ ]:=
```
x1 = Dt3Z ** v - 2 (p + 2) (h - 3 p + 6) (p + 1) ^ (-1) sh[-3, -1] ** sh[3, 1] ** v /.
   {sh[-3, -1] ** sh[3, 1] ** v → (Z13 - (2 (p + 2)) ^ (-1) Z12 ** Z23) ** Z31 ** v}
```

Out[ ]=
$$8\,i\,\text{CKi} ** v - 2\,\text{CKi} ** \text{CKi} ** v + 2\,\text{CKi} ** \text{WW0} ** v + 24\,\text{Z13} ** \text{Z31} ** v + 24\,\text{Z23} ** \text{Z32} ** v - \frac{1}{1+p} 2 \times (6 + h - 3\,p) \times (2 + p) \left(\text{Z13} ** \text{Z31} ** v - \frac{1}{2}\,\frac{2\,\text{Z13} + \text{Z23} ** \text{Z12}}{2+p} ** \text{Z31} ** v\right) - \frac{1}{9}\,i\,\text{CKi} ** \text{CKi} ** \text{CKi} ** v + i\,\text{CKi} ** \text{WW0} ** \text{WW0} ** v + i\,\text{Z12} ** \text{CKi} ** \text{Z21} ** v + 2\,i\,\text{Z13} ** \text{CKi} ** \text{Z31} ** v - 6\,i\,\text{Z13} ** \text{WW0} ** \text{Z31} ** v - 6\,\text{Z13} ** \text{Z21} ** \text{Z32} ** v + 2\,i\,\text{Z23} ** \text{CKi} ** \text{Z32} ** v + 6\,i\,\text{Z23} ** \text{WW0} ** \text{Z32} ** v + 6\,\text{Z23} ** \text{Z12} ** \text{Z31} ** v$$

In[ ]:=
```
x2 = x1 //. { CKi ** v → -I h v, WW0 ** v → -I p v, XX_ ** (ff_ v) :→ ff XX ** v, Z21 ** v → 0,
      (ff_^(-1) XX_) ** YY_ :→ ff^(-1) XX ** YY} /. Z12 ** Z31 ** v → -(2 (p + 1)) Z32 ** v // Simplify
```

Out[ ]=
$$8\,h\,v + 2\,h^2\,v + \frac{h^3\,v}{9} - 2\,h\,p\,v - h\,p^2\,v - \frac{2\,(h + 3 \times (4 + p))\,\text{Z23} ** ((1 + p)\,\text{Z32} ** v)}{1+p} - 2 \times (-6 + h - 3\,p)\,\text{Z13} ** \text{Z31} ** v + 24\,\text{Z23} ** \text{Z32} ** v + 2\,i\,\text{Z13} ** \text{CKi} ** \text{Z31} ** v - 6\,i\,\text{Z13} ** \text{WW0} ** \text{Z31} ** v - 6\,\text{Z13} ** \text{Z21} ** \text{Z32} ** v + 2\,i\,\text{Z23} ** \text{CKi} ** \text{Z32} ** v + 6\,i\,\text{Z23} ** \text{WW0} ** \text{Z32} ** v$$

In[ ]:=
```
x3 = x2 //. {WW0 ** XX_ ** v :→ -I p XX ** v + lb[WW0, XX] ** v,
     CKi ** XX_ ** v :→ -I h XX ** v + lb[CKi, XX] ** v, XX_ ** (p YY_) :→ p XX ** YY,
     XX_ ** (h YY_) :→ h XX ** YY, Z21 ** XX_ ** v :→ 0 + lb[Z21, XX] ** v} // Simplify
```

Out[ ]=
$$\frac{1}{9 \times (1+p)} (h + 3 \times (4 + p))\,(h\,(6 + h - 3\,p) \times (1 + p)\,v - 18\,\text{Z23} ** (p\,\text{Z32} ** v) + 18\,p\,\text{Z23} ** \text{Z32} ** v)$$

In[ ]:=
```
x3 //. {XX_ ** (p YY_) :→ p XX ** YY} // Simplify // Factor
```

Out[ ]=
$$\frac{1}{9}\,h\,(6 + h - 3\,p) \times (12 + h + 3\,p)\,v$$