Analysis in one complex variable
Lecture 1 – Introduction

Gil Cavalcanti

Utrecht University

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Utrecht

The complex numbers

- Invented by Gerolano Cardano in the 1500's.

- They come about when trying to find solutions to polynomial equations.

The complex numbers

- Add to $\mathbb{R}$ the solution to $x^2 + 1 = 0$: now $\sqrt{-1}$ exists!
- But where is it?

Complex Algebra

- We can manipulate $\sqrt{-1}$ formally:

$$(x+\sqrt{-1}y)(x'+\sqrt{-1}y') = (xx'-yy')+\sqrt{-1}(xy'+x'y)$$
$$= (x'+\sqrt{-1}y')(x+\sqrt{-1}y).$$

- If $z=(x+\sqrt{-1}y)$, the complex conjugate of z is $\bar{z}=x-\sqrt{-1}y$.
- We have $z\bar{z}=x^2+y^2=\|z\|^2$, hence if $z\neq 0$, $z^{-1}=\frac{\bar{z}}{\|z\|^2}$.
- Hence we can divide by complex numbers!

$$zw^{-1}=z\frac{\bar{w}}{\|w\|^2}$$

Complex Algebra

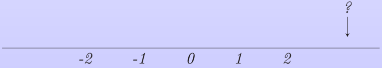
- But be aware

$$1 = \sqrt{1} = \sqrt{(-1)^2} \neq (\sqrt{-1})^2 = -1$$

- This problem ultimately comes from the fact that $\sqrt{\bullet}$ is not a function.
- Introduce $i = \sqrt{-1}$.

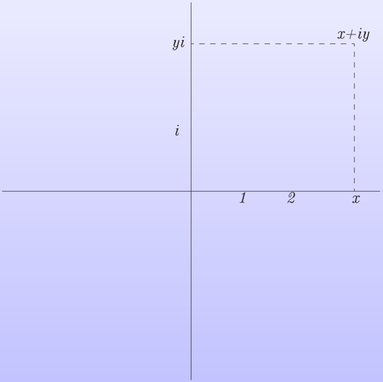
Where is *i*?

- One of the axioms for the real line is that it is complete: it has no holes.



Where is *i*? – A new dimension

- To introduce i we need an *extra dimension*.



- Geometric interpretation upshot: polar coordinates: $z = \|z\|(\cos\theta + i\sin\theta)$.

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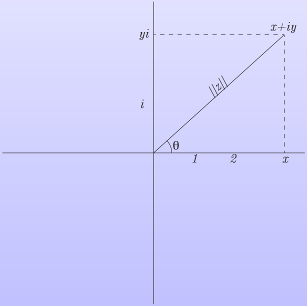

Calculus – Derivatives

- For $f : \mathbb{R} \to \mathbb{R}$ we have

$$\left.\frac{df}{dx}\right|_{x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- Can do the same for a complex function, $f : \mathbb{C} \to \mathbb{C}$.

$$\left.\frac{df}{dz}\right|_{z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

- But $\mathbb{C} = \mathbb{R}^2$! Any relation between derivatives?

Calculus – Integration

- Less obvious.
- Integrate over paths?
- Integrate over domains?
- What should be the answer?

$$\int_{?} f = ?$$

- Should integration and differentiation be "inverses" of each other?

Calculus – Why?

Because we can!

Theorem (Fundamental Theorem of Algebra)

Every complex polynomial of positive degree has a root.

Complex functions turn out to be very useful in analysis.

Transforms

- The Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ is defined as

$$\hat{f} : \mathbb{R} \to \mathbb{R}, \qquad \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx.$$

- Basic property (integration by parts):

$$(\hat{f'})(\xi) = 2\pi i \xi \hat{f}(\xi).$$

  Changes differential equations into polynomial equations (easier to solve/tackle).
- $u_{tt} = u_{xx} \rightsquigarrow \hat{u}_{tt} = (2\pi i \xi)^2 \hat{u} \rightsquigarrow \hat{u} = a e^{2\pi i \xi t} + b e^{-2\pi i \xi t}$
- Once we have $\hat{u}$, how do we get u?

Rerun of real analysis?

- Having complex derivatives is much more restrictive than having real derivatives.
- If $f : \mathbb{C} \to \mathbb{C}$ has complex derivatives at all points, then f has infinitely many derivatives and its Taylor series converges to f.

Exercises

Exercise

Let $z = r(\cos\theta + i\sin\theta)$ *and* $w = \rho(\cos\tau + i\sin\tau)$. *Show that*

$$zw = r\rho(\cos(\theta+\tau) + i\sin(\theta+\tau)).$$

Conclude that $(cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.