Analysis in one complex variable
Lecture 1 – Complex numbers

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The complex numbers

- $\mathbb{C}$ is obtained from $\mathbb{R}$ by adding a root, i, to the equation $x^2 + 1 = 0$.
- A complex number is always of the form $z = x + iy$, with $x, y \in \mathbb{R}$.
- x is the real part of z, $x = \mathrm{Re}(z)$
- y is the imaginary part of z, $y = \mathrm{Im}\,(z)$.
- We can add complex numbers componentwise:

$$(x + iy) + (x' + iy') = (x + x') + i(y + y').$$

- We can multiply complex numbers using $i^2 = -1$:

$$(x + iy)(x' + iy') = (xx' - yy'') + i(xy' + yx').$$

The complex numbers

- the complex conjugate of $z = x + iy$ is

$$\bar{z} = x - iy$$

- z is real iff $\bar{z} = z$
- z is imaginary iff $\bar{z} = -z$

The complex numbers

- Basic properties:

$$\begin{aligned}
\bar{\bar{z}} &= z \\
\overline{zw} &= \bar{z}\bar{w} \\
z\bar{z} &= (x^2 + y^2) = \|z\|^2 \\
z^{-1} &= \frac{\bar{z}}{\|z\|^2} \\
\|\bar{z}\| &= \|z\| \\
\|zw\| &= \|z\|\|w\|
\end{aligned}$$

$$\|zw\|^2 = zw\overline{zw} = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = \|z\|^2\|w\|^2.$$

The complex numbers

- We have

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}), \qquad \mathrm{Im}\,(z) = \frac{1}{2i}(z - \bar{z}).$$

The complex numbers

- If we use complex addition and restrict multiplication to $\mathbb{R} \subset \mathbb{C}$ we see that $\mathbb{C} = \mathbb{R}^2$ as a vector space.
- The only new thing is multiplication by i.

The complex numbers - Polar coordinates

- We write a complex number $z = x + iy$ in polar coordinates:

$$z = r(\cos\theta + i\sin\theta),$$

with $r = \|z\|$, $\cos\theta = \frac{x}{\|z\|}$, $\sin\theta = \frac{y}{\|z\|}$.

- Complex multiplication has a simple description:

$$m_z \colon \mathbb{C} \to \mathbb{C}, \qquad m_z(w) = zw,$$

corresponsds to scaling by $\|z\|$ and rotation by the angle θ.

- The angle θ is the *argument* of z.

The complex numbers - Polar coordinates

- If we focus on the unitary circle, that is, numbers of the form

  $$z = \cos\theta + i\sin\theta, \qquad w = \cos\varphi + i\sin\varphi,$$

  then

  $$zw = \cos(\theta + \varphi) + i\sin(\theta + \varphi),$$

  Writing a number in polar coordinates transforms multiplication into sums.

The complex numbers - Polar coordinates

- $\frac{d(\cos t + i\sin t)}{dt} = -\sin t + i\cos t = i(\cos t + i\sin t)$.
- For $a \in \mathbb{R}$ the solution to $\frac{df}{dt} = af$ is given by $f(t) = e^{at}$. So we *define*
$$e^{i\theta} := \cos\theta + i\sin\theta.$$

- Properties rephrased: $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$ and $\frac{d}{dt}e^{it} = ie^{it}$.
- We write in general $z = re^{i\theta}$.

The complex numbers - Polar coordinates

- We define further $e^{x+iy} := e^x e^{iy}$
- It follows that $e^z e^w = e^{z+w}$.

The complex numbers - Complex functions

Definition

A complex function is a function with values in $\mathbb{C}$:

$$f\colon S \to \mathbb{C}$$

- Here we will focus on the case when $S \subset \mathbb{C}$.
- Composing with Re and Im we obtain underlying real functions:

$$u = \mathrm{Re}(f), \qquad v = \mathrm{Im}\,(f), \qquad f = u + iv$$

The complex numbers - Complex functions

Example

- $f(z) = \|z\|^2 = z\bar{z}$,
- $f(z) = \mathrm{Re}(z) = \frac{z+\bar{z}}{2}$,
- $f(z) = z^2 + 2z + 1$,
- $f(z) = e^z$,
- $m_z(w) = zw$ for a fixed $z \in \mathbb{C}$.