Analysis in one complex variable
Lecture 1 – Complex derivative

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April 2020
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The complex derivative

Definition

The *complex derivative* of a continuous function $f\colon S \subset \mathbb{C} \to C$ defined on an open set $S \in \mathbb{C}$ at a point $z_0 \in S$, is the limit

$$\frac{df}{dz}(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

The function f is *holomorphic in S* if it has complex derivative at every $z \in S$.

The notion of convergence comes from the Euclidean distance in $\mathbb{R}^2$.

Examples and non examples

- Any constant function is complex differentiable and has zero derivative.
- The identity function is complex differentiable and has derivative 1:
$$\lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1.$$

Lemma (Leibniz rule)

If $f, g \colon S \to \mathbb{C}$ both have complex derivative at z then their product also has a complex derivative at z:

$$\frac{d}{dz}(gf)(z) = g(z)\frac{df}{dz}(z) + \frac{dg}{dz}(z)f(z)$$

Examples and non examples

Example

Given $n \in \mathbb{N}$, the function $z \mapsto z^n$ is holomorphic and its derivative is

$$\frac{dz^n}{dz} = nz^{n-1}.$$

Indeed, we already computed the derivative of the identity map and of products of maps. The result follows by induction on n.

Example

Any complex polynomial is holomorphic.

Examples and non examples

Lemma

If $f, g\colon S \to \mathbb{C}$ both have complex derivative at z and $g(z) \neq 0$, then f/g has a complex derivative at z:

$$\frac{d}{dz}\left(\frac{f}{g}\right)(z) = \frac{g(z)\frac{df}{dz}(z) - \frac{dg}{dz}(z)f(z)}{g(z)^2}.$$

Example

The quotient of to polynomials, f/g is a holomorphic function in the set $S = g^{-1}(\mathbb{C}\backslash\{0\}) \subset \mathbb{C}$.

Examples and non examples

Lemma (Chain rule)

If $f\colon S \to \mathbb{C}$ and $g\colon S' \to \mathbb{C}$ are such that $g(S') \subset S$, g has complex derivative at z_0 and f has complex derivative at $g(z_0)$, then $f \circ g\colon S' \to \mathbb{C}$ has derivative at z_0 given by

$$\frac{d}{dz}(f \circ g)(z_0) = \frac{df}{dz}(g(z_0))\frac{dg}{dz}(z_0).$$

Examples and non examples

Example

Consider $f\colon \mathbb{C} \to \mathbb{C}, f(z) = \bar{z}$. We claim that this function is not complex differentiable.
Compute it by taking specific paths approaching z_0.
Taking the path $z_0 + t$, with $t \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ we get

$$\lim_{t\to 0} \frac{\overline{z_0 + t} - \overline{z_0}}{t} = \lim_{t\to 0} \frac{t}{t} = 1.$$

While taking the path $z_0 + it$, with $t \in (-\varepsilon, \varepsilon)$ we get

$$\lim_{t\to 0} \frac{\overline{z_0 + it} - \overline{z_0}}{it} = \lim_{t\to 0} \frac{-it}{it} = -1.$$

So the limit does not exist.

Examples and non examples

Example

The functions Re and Im are not complex differentiable.
Indeed, we have, for example

$$\bar{z} = -z + 2\mathrm{Re}(z),$$

so if Re had a complex derivative, so would conjugation.

Cauchy–Riemann equations

These previous examples show that being differentiable as a real function is not enough. To be complex differentiable seems to force a relation between partial derivatives of f.

Lemma (Cauchy–Riemann equations)

Let $f = u + iv \colon S \subset \mathbb{C} \to C$ be a function. If f is complex differentiable at $z_0 \in S$ we have

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0).$$

Cauchy–Riemann equations

Proof.

Just as for $\bar{z}$, we assume that f is complex differentiable and compute its limit using two different paths.
First use the path $t \mapsto z + t$, for $t \in (-\varepsilon, \varepsilon)$:

$$\begin{aligned}\frac{df}{dz}(z) &= \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} \\ &= \lim_{t\to 0} \frac{u(x+t+iy) - u(x+iy) + i(v(x+t+iy) - v(x+iy))}{t} \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).\end{aligned}$$

□

Cauchy–Riemann equations

Proof.

Next we use the path $t \mapsto z + it$, for $t \in (-\varepsilon, \varepsilon)$:

$$\begin{aligned}
\frac{df}{dz}(z) &= \lim_{t \to 0} \frac{f(z+it) - f(z)}{it} \\
&= \lim_{t \to 0} \frac{u(x+i(y+t)) - u(x+iy) + i(v(x+i(y+t)) - v(x+iy))}{it} \\
&= -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y).
\end{aligned}$$

Comparing real and imaginary parts of the two expressions we obtain the relations. □

Cauchy–Riemann equations

Rephrasing, if $f = u + iv$ is holomorphic, and we regard it as a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, then its derivative is of the form

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}.$$

Cauchy–Riemann equations

Theorem

Let $f : S \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous function with continuous partial derivatives. If the derivative of f satisfies the Cauchy-Riemann equations, then f is holomorphic using the identification $\mathbb{R}^2 = \mathbb{C}$.

Cauchy–Riemann equations

Proof.

Since the partial derivatives of f exist and are continuous, f is differentiable, that is, for every z_0

$$f(z) = f(z_0) + Df|_{z_0}(z - z_0) + o(z - z_0),$$

where

$$\lim_{z \to z_0} \frac{\|o(z - z_0)\|}{\|z - z_0\|} = 0.$$

□

Cauchy–Riemann equations

Proof.

Since f satisfies the Cauchy–Riemann relations,

$$Df|_{z_0} = \begin{pmatrix} u_x|_{z_0} & -v_x|_{z_0} \\ v_x|_{z_0} & u_x|_{z_0} \end{pmatrix}$$

and we see that under the identification $\mathbb{R}^2 = \mathbb{C}$ we have $Df|_{z_0}(z - z_0) = (u_x + iv_x)|_{z_0}(z - z_0)$.
Therefore

$$\frac{f(z) - f(z_0)}{z - z_0} = u_x|_{z_0} + iv_x|_{z_0} + \frac{o(z - z_0)}{z - z_0}.$$

And f is complex differentiable at z_0 with derivative $u_x|_{z_0} + iv_x|_{z_0}$. □