Analysis in one complex variable
Lecture 2 – Complex derivative II

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Last lecture: Cauchy–Riemann equations

Recall

Theorem

Let $f : S \subset \mathbb{C} \to \mathbb{C}$ be a function and $z \in S$ and write it in terms of its real and imaginary parts: $f = u + iv$. Then f is holomorphic at z if and only if the derivative of f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ at z exists and satisfies the Cauchy–Riemann equations

$$u_x|_z = v_y|_z \qquad u_y|_z = -v_x|_z.$$

Last lecture: Cauchy–Riemann equations

Example

Consider $f(z) = \bar{z}$. Then, as a function in $\mathbb{R}^2$ we have

$$f(x, y) = (x, -y),$$

so $u_x = 1$, $u_y = 0$, $v_x = 0$ and $v_y = -1$.
We see that $u_x \neq v_y$ hence this function is not holomorphic.

Last lecture: Cauchy–Riemann equations

- The Cauchy-Riemann equations phrase a new property (holomorphicity) in familar terms (differentiability of functions of many variables).
- They have *computational value*, but do not shed little light into the holomorphic condition.
- What would be the complex derivative of a function $f\colon \mathbb{C}^n \to \mathbb{C}^m$?

Linear algebra

Consider $m_z \colon \mathbb{C} \to \mathbb{C}$. We see that it is a real linear transformation:

$$m_z(a + \lambda b) = z(a + \lambda b) = za + z\lambda b = m_z(a) + \lambda m_z(b).$$

Linear algebra

Determine the matrix for m_z for the usual basis of $\mathbb{R}^2$, $\{1, i\}$:
If $z = x + iy$, then

$$m_z(1) = x + iy \qquad m_z(i) = -y + ix$$

Hence, as a matrix,

$$m_z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Compare this to the CR-equations.

Cauchy–Riemann equations v2

Theorem

Let $f : S \subset \mathbb{C} \to \mathbb{C}$ be a function and $z \in S$. Then f is holomorphic at z if and only if the real derivative of f at z exists and corresponds to multiplication by a complex number.

Any ideas of what it means for $f : \mathbb{C}^n \to \mathbb{C}^m$ to be complex differentiable?

Linear algebra

The only difference between $\mathbb{R}^2$ and $\mathbb{C}$ is that the latter has the operation "multiplication by i".
We have

$$m_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Lemma

A linear transformation $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ corresponds to complex multiplication by some number $z \in \mathbb{C}$ if and only if A commutes with m_i.

Linear algebra

Proof.

We perform a direct computation. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$m_i \circ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

$$A \circ m_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}.$$

Hence $m_i \circ A = A \circ m_i$ iff $a = d$ and $c = -b$. □

Cauchy–Riemann equations v3

Theorem

Let $f : S \subset \mathbb{C} \to \mathbb{C}$ be a function and $z \in S$. Then f is holomorphic at z if and only if the real derivative of f at z exists and commutes with multiplication by i:

$$df \circ m_i = m_i \circ df.$$

Definition

Let $f : S \subset \mathbb{C}^n \to \mathbb{C}^m$ be a function and $z \in S$. Then f is *holomorphic* at z if and only if the real derivative of f at z exists and corresponds to matrix in $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m) \subset \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2m})$ or, equivalently, if df commutes with multiplication by i:

$$df \circ m_i = m_i \circ df.$$

The pedantic stuff

- Does m_i make sense?
- Originally m_i was defined as a linear transformation $\mathbb{C} \to \mathbb{C}$, but derivatives should be applied to tangent vectors.
- Notice that m_i induces an operation on $T_z\mathbb{C}$ as follows.

The pedantic stuff

- We normally distinguish the linear transformation m_i from the induced operation on tangent spaces, which denote by I.
- So f is holomorphic iff $df \circ I = I \circ df$.