Analysis in one complex variable
Lecture 2 – Conformal structure

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Linear algebra is the best!

Yes, it is.

Inner product

Definition

An *inner product* on a vector space V is a bilinear map

$$\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{R}$$

which is symmetric and positive definite, that is

$$\langle X, Y \rangle = \langle Y, X \rangle, \qquad \text{and} \qquad \langle X, X \rangle > 0 \text{ if } X \neq 0.$$

Inner product

Lemma (Cauchy–Schwarz inequality)

Given a vector space with inner product, $(V, \langle \cdot, \cdot \rangle)$*,* V *we have*

$$\langle X, Y \rangle^2 \leq \langle X, X \rangle \langle Y, Y \rangle.$$

Inner product

Proof.

$$
\begin{aligned}
0 &\leq \langle Y - \frac{\langle X, Y\rangle}{\langle X, X\rangle} X, Y - \frac{\langle X, Y\rangle}{\langle X, X\rangle} X\rangle \\
&= \langle Y, Y\rangle - 2\frac{\langle X, Y\rangle^2}{\langle X, X\rangle} + \frac{\langle X, Y\rangle^2}{\langle X, X\rangle} \\
&= \langle Y, Y\rangle - \frac{\langle X, Y\rangle^2}{\langle X, X\rangle}.
\end{aligned}
$$

Rearranging we obtain the result. □

Norm

- Given a vector space with inner product, $(V, \langle\cdot,\cdot\rangle)$, we define
$$\|X\| = \sqrt{\langle X, X\rangle}.$$
- $\|\cdot\|$ is a norm, that is,
  1. $\|X\| > 0$, if $X \neq 0$,
  2. $\|\lambda X\| = |\lambda|\|X\|$,
  3. $\|X + Y\| \leq \|X\| + \|Y\|$.

For the last:

$$\begin{aligned}\|X+Y\|^2 &= \langle X+Y, X+Y\rangle = \langle X, X\rangle + 2\langle X, Y\rangle + \langle Y, Y\rangle \\ &\leq \langle X, X\rangle + 2\sqrt{\langle X, X\rangle\langle Y, Y\rangle} + \langle Y, Y\rangle \\ &= (\sqrt{\langle X, X\rangle} + \sqrt{\langle Y, Y\rangle})^2\end{aligned}$$

Angle

- From the Cauchy–Schwarz inequality

$$\langle X, Y \rangle \leq \|X\|\|Y\|$$

there is a unique $\theta \in [0, \pi]$ such that

$$\langle X, Y \rangle = \|X\|\|Y\| \cos\theta.$$

This is the *unoriented angle* between X and Y ($\theta = \measuredangle(X, Y)$)

Angle

If we consider $\mathbb{R}^2$ with its usual *orientation* we can further assign a sign to $\measuredangle(X, Y)$:

$$\measuredangle(X, Y) > 0 \text{ if } \{X, Y\} \text{ is a positive basis for } \mathbb{R}^2$$

$$\measuredangle(X, Y) < 0 \text{ if } \{X, Y\} \text{ is a negative basis for } \mathbb{R}^2$$

Ambiguous for angles $\pm\pi$.
Fix this by declaring $\measuredangle \in (-\pi, \pi]$.

Angle

- If $\langle \cdot, \cdot \rangle$ is an inner product, so is $\langle\langle \cdot, \cdot \rangle\rangle := \lambda \langle \cdot, \cdot \rangle$ for $\lambda > 0$.
- $\langle X, Y \rangle = \|X\| \|Y\| \cos\theta$.
- For these inner products we have

$$\measuredangle_{\langle \cdot, \cdot \rangle} = \measuredangle_{\langle\langle \cdot, \cdot \rangle\rangle}$$

Definition

A *conformal structure* on a vector space is an inner product up to scaling on V.

A conformal structure is a way to measure angles.

Angle

Definition

Given a vector space V with a conformal structure, a *conformal transformation* of V is a linear transformation $A \colon V \to V$ that preserves angles.

Lemma

Given a conformal transformation A on V and an inner product which determines the conformal structure on V, there is a unique orthogonal transformation P and $r > 0$ such that $A = rP$.

Angle

Proof.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V.
Then $\{Ae_1, \ldots, Ae_n\}$ is a collection of pairwise orthogonal vectors in V.
We claim that $\|Ae_i\| = \|Ae_j\|$ for all i, j.
Indeed, $e_i + e_j$ and $e_i - e_j$ are orthogonal to each other, hence so are $A(e_i + e_j)$ and $A(e_i - e_j)$:

$$0 = \langle A(e_i + e_j), A(e_i - e_j)\rangle = \langle A(e_i), A(e_i)\rangle - \langle A(e_j), A(e_j)\rangle.$$

Let $r = \|Ae_1\|$. Then $P = \frac{A}{r}$ is orthogonal. $\square$

Angles in $\mathbb{R}^2$

From Linear Algebra/Group Theory we know that given $A \in O(2)$ there is $\theta \in (-\pi, \pi]$ such that

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Angles in $\mathbb{R}^2$

Given a conformal transformation A of $\mathbb{R}^2$ there are $x, y \in \mathbb{R}$ such that $x^2 + y^2 \neq 0$ and

$$A = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \qquad \text{or} \qquad A = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$$

A preserves orientations (and oriented angles) if and only if it is of the first type.

Angles in $\mathbb{R}^2$

Lemma

A linear transformation of $\mathbb{R}^2$ preserves oriented angles if and only if it corresponds to multiplication by a complex number under the identification $\mathbb{R}^2 = \mathbb{C}$.

Holomorphic functions and angles

Proposition

A function $f : S \subset \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if it has a real derivative and df preserves oriented angles whenever it is not zero.

Exercise

Exercise (Artwork)

In $\mathbb{C}$ consider the collection of lines

$$S = \{x + iy \colon x \text{ or } y \in \mathbb{Z}\}.$$

Draw $f(S)$ for the following functions:

- $f(z) = z^2$,
- $f(z) = z^3$,
- $f(z) = e^z$.