Analysis in one complex variable
Lecture 2 – Power Series

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Algebra is the best!

Kidding, of course it is not.

Formal Power Series

- A formal power series is an algebraic gadget.
- Given any ring R, we can consider the formal power series in one variable in R, denoted by $R[[x]]$.
- Elements of $R[[x]]$ are of the form $a = \sum_{n=0}^{\infty} a_i x^i$, with $a_i \in R$.
- $R[[x]]$ is a purely algebraic object.
- We can add, multiply, invert and even compose power series (sometimes), but we can not evaluate them at a point $x \in R$.

Formal Power Series

$$(\sum a_i x^i) + (\sum b_i x^i) = \sum (a_i + b_i) x^i$$

$$(\sum a_i x^i)(\sum b_i x^i) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} (a_j b_{i-j}) x^i$$

In particular, for $a \in R[[x]]$, $a^2, a^3, \cdots$ are also in $R[[x]]$.

$$(\sum_{i=0}^{\infty} a_i x^i) \circ (\sum_{j=1}^{\infty} b_j x^j) = \sum_{i=0}^{\infty} a_i (\sum_{j=1}^{\infty} b_j x^j)^i$$

Notice that because the sum $\sum_{j=1}^{\infty} b_j x^j$ starts at $j = 1$, the expression on the right has finitely many contributions to the coefficient of x^i for any fixed i.

Formal Power Series

Definition

The *order* of a power series a, $\mathrm{ord}(a)$, is the index of the first nonzero coefficient in a

If R is a field (e,.g. $\mathbb{R}$), we have that $\mathrm{ord}(a\alpha) = \mathrm{ord}(a) + \mathrm{ord}(\alpha)$. In particular, if a has an inverse, α, we have that

$$0 = \mathrm{ord}(1) = \mathrm{ord}(a\alpha) = \mathrm{ord}(a) + \mathrm{ord}(\alpha)$$

hence both a and α must have order 0.

Lemma

A formal power series is invertible iff it has order zero.

Formal Power Series

Definition

Two power series, a and b *agree modulo* x^n if $a_i = b_i$ for all $i < n$. We denote this by

$$a \equiv b \bmod x^n.$$

Convergent Power Series

- Given a power series $a \in \mathbb{R}[[x]]$ or $\mathbb{C}[[x]]$, it becomes a valid question to ask with which real or complex numbers we can substitute x and get a convergent series.
- We are led to the world of convergent series/sequences as well as convegent series/sequence of functions.

Convergent Power Series

Definition

A series $\sum a_i$ with $a_i \in \mathbb{C}$ *converges absolutely* if $\sum |a_i|$ converges.

Definition

A sequence of complex functions $\{f_n\}$ *converges uniformly* if there is a complex function f with the property that for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\|f_n - f\| < \varepsilon$ for all $n > n_0$. Here $\|\cdot\|$ is the supremum norm of the function inside.

Convergent Power Series

Theorem

Given $a = \sum a_i x^i \in \mathbb{C}[[x]]$*, there is* $r_0 \in [0, \infty]$ *such that the series* $\sum a_i z^i$ *converges uniformly and absolutely in* D_r*, the disc of radius* r*, for every* $r < r_0$ *and diverges for all* z *with* $|z| > r$*.*

Definition

The number r_0 above is the *radius of convergence* of the power series.

Convergent Power Series

Lemma

Let $a = \sum a_n z^n$ *be a power series and* r *be its radius of convergence. Then*

$$\frac{1}{r} = \limsup |a_n|^{1/n}.$$

Convergent Power Series

Proof.

Let $t = \limsup |a_n|^{1/n}$. We will only do the case $0 < t < \infty$.
For any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|a_n| < (t+\varepsilon)^n$,for $n > n_0$, by definition of $\limsup$.
If $|z| < 1/(t+\varepsilon)$, say $|z| = |1/(t+\varepsilon+\delta)$ then

$$|a_n z^n| < (t+\varepsilon)^n \frac{1}{(t+\varepsilon+\delta)^n} = \left(\frac{t+\varepsilon}{t+\varepsilon+\delta}\right)^n$$

hence $\sum |a_n z^n|$ converges by comparing with the geometric series.
Therefore, the radius of converge, r, is bigger than $1/(t+\varepsilon)$ for every ε. Hence $r \geq \frac{1}{t}$

Convergent Power Series

Proof.

Conversely, given $\varepsilon > 0$, there is an infinite set $S \subset \mathbb{N}$ such that for all $n \in S$, $|a_n| > (t - \varepsilon)^n$, by definition of $\limsup$.
If $|z| > \frac{1}{t-\varepsilon}$, then for every $n \in S$,

$$|a_n z^n| > (t - \varepsilon)^n \frac{1}{(t - \varepsilon)^n} = 1,$$

hence the series does not converge as the general term does not converge to 0.
Therefore $r < \frac{1}{t-\varepsilon}$ for all $\varepsilon > 0$. Hence $r \leq \frac{1}{t}$. □

Convergent Power Series

Corollary

Given a series $a = \sum a_n z^n$*, if* $t = \lim |a_n|^{1/n}$ *exists, then* $r = 1/t$*.*

Formal vs Convergent Power Series

Lemma

The operations of sum, product, inversion and composition of functions agree with the same operations for power series.

Analytic functions

Definition

A function $f : S \subset \mathbb{R}^n \to \mathbb{R}^m$ is *analytic* if for every point $x \in S$ the power series expansion of f centered at x has positive radius of convergence and converges to f.

Remark

If the power series expansion of f has positive radius of convergence, say it converges on $D_r(x)$, the disc of radius r around x, then for any $x' \in D_r(x)$ the corresponding series also converges at least in the disc of radius $r - |x - x'|$.
In particular, if the power series expansion of f at x has infinite radius of convergence, then f is analytic.

Analytic functions

Example

We are familiar with the expressions

$$e^x = \sum \frac{x^n}{n!}$$

$$\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{1}{1-x} = \sum x^n, \qquad \text{for } |x| < 1.$$

showing that each of these functions is analytic.

Analytic functions

Example

The series $\sum n!x^n$ has zero radius of convergence since $\lim(n!)^{1/n} = \infty$.
The function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

has infinitely many derivatives at zero and they all vanish. Hence, the power series expansion of f at 0 converges, but does not converge to f.

Derivatives of analytic functions

Theorem

If $f(z) = \sum a_n z^n$ with $z \in \mathbb{C}$ has radius of convergece r, then on the interior of D_r f is holomorphic and its derivative is

$$\frac{df}{dz} = \sum n a_n z^{n-1},$$

which also has radius of convergence r.

Exercise

Exercise

Let $f : \mathbb{C} \to \mathbb{C}$ *be given by*

$$f(z) = \sum \frac{z^n}{n!}.$$

Show that

- f *has infinite radius of convergence*
- *for* $x, y \in \mathbb{R}$ $f(x) = e^x$ *and* $f(iy) = \cos y + i \sin y$
- $f(z + w) = f(z)f(w)$.