Analysis in one complex variable
Lecture 3 – Inverse map theorem

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April 2020
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Last lecture: Power series

Recall:

- Formal power series,
- Sums, products, multiplicative inverses ($1/f$) and composition,
- Convergent power series and *analytic functions*
- Derivatives.
  In particular

$$f = \sum a_n z^n \Rightarrow \frac{d^n f}{dz_n}(0) = n! a_n.$$

Functional inverse

Given $f\colon \mathbb{C} \to \mathbb{C}$, analytic when does it have an analytic (right) inverse?

$$g\colon \mathbb{C} \to \mathbb{C}, \qquad f \circ g(z) = z, \text{ for all } z.$$

Functional inverse

Theorem (Inverse Function Theorem)

The following hold:

- *Let* $f = \sum_{i=1}^{\infty} a_i z^i$ *be a formal power series. If* $a_1 \neq 0$*, there is a formal power series* $g = \sum_{i=1}^{\infty} b_i z^i$ *such that*

$$f \circ g(z) = z$$

  further $g \circ f(z) = z$.
- *If* f *is a convergent power series, so is* g.
- *Suppose that* $f : U \subset \mathbb{C} \to \mathbb{C}$ *is an analytic function in an open set* U *and* $z_0 \in U$*. If* $\frac{df}{dz}(z_0) \neq 0$*, then* f *is a local analytic isomorphism in a neighbourhood of* z_0.

Functional inverse

Proof.

Claim 1: We simply construct g term by term.
Degree 0: Since both f and g have vanishing constant terms, it follows that no matter which further choices $f \circ g$ will have vanishing constant term.

Functional inverse

Proof.

Degree 1: Write the series for $f \circ g$ module z^2. It is just

$$\sum_{i=1}^{\infty} a_i \left(\sum_{j=1}^{\infty} b_j z^j\right)^i = a_1 b_1 z \qquad \bmod z^2$$

So if we choose $b_1 = 1/a_1$ further choices of values for b_j with $j > 1$ will not affect the equality

$$f \circ g(z) = z \qquad \bmod z^2$$

Functional inverse

Proof.

Degree 2: Write the degree two term of $f \circ g$:

$$(\sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} b_j z^j)^i)_2 = (a_1(b_1 z^1 + b_2 z^2)^1 + a_2(b_1 z^1 + b_2 z^2)^2)_2$$
$$= (a_1 b_2 + a_2 b_1^2) z^2.$$

So if we choose $b_2 = -a_2 b_1^2 / a_1$ further choices of values for b_j with $j > 2$ will not affect the equality

$$f \circ g(z) = z \qquad \mod z^3$$

Functional inverse

Proof.

Degree 3: Write the degree three term of $f \circ g$:

$$(\sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} b_j z^j)^i)_3 = (a_1(b_1 z^1 + b_2 z^2 + b_3 z^3)^1 + a_2(b_1 z^1 + b_2 z^2 + b_3 z^3)^2 +$$

$$= (a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3) z^3.$$

So if we choose $b_3 = -(2a_2 b_1 b_2 + a_3 b_1^3)/a_1$ further choices of values for b_j with $j > 3$ will not affect the equality

$$f \circ g(z) = z \qquad \mod z^4$$

Functional inverse

Proof.

General case: Write the degree k term of $f \circ g$:

$$(\sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} b_j z^j)^i)_k = a_1 b_k + P_k(a_1, a_2, \ldots a_k, b_1, \ldots, b_{k-1})$$

where P_k is a polynomial with positive integral coefficients (arising from binomial coefficients).
Set

$$b_k = -\frac{P_k(a_1, a_2, \ldots a_k, b_1, \ldots, b_{k-1})}{a_1}$$

Then

$$f \circ g(z) = z \qquad \mod z^{k+1}$$

Functional inverse

Proof.

Claim: $g \circ f(z) = z$.
Indeed, by first part, $g = \sum_{i=1}^{\infty} b_i z^1$ with $b_1 \neq 0$, hence also has a right inverse, h. Then

$$f(z) = f \circ (g \circ h)(z) = (f \circ g) \circ h(z) = h(z).$$

Functional inverse

Proof.

Claim 2: If f converges, then g converges.
This step is always very important.
This step is always annoying and requires a trick or perseverance.
Today we do a trick.

Functional inverse

Proof.

Claim 3. By using translations, assume that $z_0 = 0$ and $f(z_0) = 0$, so

$$f(z) = \sum_{i=1}^{\infty} a_i z^i.$$

By the previous claims there is g analytic

$$g(z) = \sum_{i=1}^{\infty} b_i z^i.$$

for which $f \circ g = g \circ f = \mathrm{Id}$. □

The trick

- Assume $a_1 = 1$.
- To prove that g converges in some region we need to show that $\limsup |b_n|^{1/n} < \infty$.
- We do so by cooking up another sequence $\tilde{b_n}$ such that for which we can prove $|b_n| \leq |\tilde{b_n}|$ and $\limsup |\tilde{b_n}|^{1/n} < \infty$.
- Since $\limsup |a_n|^{1/n} < \infty$, there is $A > 0$ such that $|a_n| < A^n$ for all n.
- Define

$$\tilde{f}(z) = z - \sum_{i \geq 2} A^i z^i = z - \frac{A^2 z^2}{1 - Az}.$$

The trick

Properties of $\tilde{f}$:

- $|a_n| \leq A^n$ for all n and
- the coefficient of z^i in $\tilde{f}$ is negative for all $i > 1$.

Let $\tilde{g}$ be the formal inverse of $\tilde{f}$. Then compare the coefficients of $\tilde{g}$ and g.

The trick

- $\tilde{b_1} = 1 = b_1$
- $\tilde{b_2} = -\tilde{a}_2\tilde{b_1}^2 = A^2 > |a_2 b_1^2| = |b_2|$.
- $$\begin{aligned} \tilde{b_3} &= -(2\tilde{a}_2\tilde{b_1}\tilde{b_2} + \tilde{a}_3 b_1^3) \\ &= 2A^2\tilde{b_2} + A^3 \\ &> |2a_2 b_1 b_2| + |a_3 b_1^2| \\ &> |2a_2 b_1 b_2 + a_3 b_1^2| \\ &= |b_3|. \end{aligned}$$

The trick

Now we only have to prove the theorem for $\tilde{f}$, which has an explicit form.
Plugging in $\tilde{g}$ in $\tilde{f}$ we must have

$$z = \tilde{f}(\tilde{g}(z)) = \tilde{g}(z) - \frac{A^2\tilde{g}(z)^2}{1 - A\tilde{g}(z)}$$

Hence

$$(A^2 + A)\tilde{g}(z)^2 - (1 - Az)\tilde{g}(z) + z = 0$$

And

$$\tilde{g}(z) = \frac{1 + Az - \sqrt{(1 + Az)^2 - 4z(A^2 + A)}}{2(A^2 + A)}$$

The trick

Now expand this last expression in power series to show that $\tilde{g}$ is analytic.