Analysis in one complex variable
Lecture 3 – Open map theorem

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Open mapping theorem

Definition

A map between topological spaces, $f\colon A \to B$, is *open* if $f(U)$ is open in B for all U open in A.

Example

If f is a local homeomorphism, then f is open.

Example

For every $k > 0$, the map $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^k$ is open.
Need to check at 0 and away from 0.
Away from 0, it is a local homeomorphism.
At 0, it maps the ball of radius r to the ball of radius r^k.

Open mapping theorem

Lemma

Let $f(z) = a_n z^n + \dots$ with $a_n \neq 0$. Then there is an analytic diffeomorphism φ of a neighbourhood of 0 such that $f \circ \varphi(z) = z^n$.

Proof.

Let $a = a_n^{1/n}$ and write

$$f(z) = a_n z^n \left(1 + \frac{a_{n+1}}{a_n} z + \dots\right) = \left(az(1 + zh(z))^{1/n}\right)^n.$$

Notice that $(1 + zh(z))^{1/n}$ is analytic for z small, say

$$(1 + zh(z))^{1/n} = 1 + z\tilde{h}(z).$$

So

$$f(z) = (az(1 + z\tilde{h}(z)))^n.$$

Open mapping theorem

Proof.

$$f(z) = (az(1 + z\tilde{h}(z)))^n.$$

Consider $\psi(z) = az(1 + z\tilde{h}(z)) = az + az^2\tilde{h}(z)$, so that

$$f(z) = \psi(z)^n$$

and IFT $\Rightarrow \varphi = \psi^{-1}$ exists and is analytic, hence

$$f(\varphi(z)) = (\psi(\varphi(z)))^n = z^n.$$

□

Open mapping theorem

Theorem

Every nonconstant analytic map is open.

Proof.

- The composition of open maps is open,
- By the previous lemma, every analytic function is locally the composition of an isomorphism and $z \mapsto z^n$,
- By the previous examples, isomorphism and $z \mapsto z^n$ are open.

$\square$

Open mapping theorem

Theorem

Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a nonconstant analytic on the open set U. Then $|f|$ has no local maximum in U.

Proof.

- Assume that $|f|$ attains a local maximum at z_0, that is $|f(z_0)| \geq |f(z)|$ for all z in some ball, B, around z_0.
- OMT $\Rightarrow f(B)$ is an open set containing $f(z_0)$.
- For $\lambda > 1$ small $\lambda f(z_0) \in f(B)$ and hence there are points in $f(B)$ with norm greater than $|f(z_0)|$

□

Open mapping theorem

Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant polynomial. Then f has a root.

Proof.

- If $f = a_0 + a_1 z + \dots a_n z^n$ had no roots, then $1/f : \mathbb{C} \to \mathbb{C}$ would be analytic.
- For $|z|$ large, the term $a_n z_n$ dominates the rest of the polynomial and goes to infinity.
- For $|z|$ large, say, $|z| > R$, $|1/f(z)| < 1/2f(0)$.
- In B_R, $|1/f|$ is continuous in a compact, hence has a maximum: $|1/f(z_0)| \geq |1/f(z)|$ for all $z \in B_R$.
- Hence $|1/f(z_0)| \geq |1/f(z)|$ for all $z \in \mathbb{C}$.

□