Analysis in one complex variable
Lecture 3 – Integration

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Curves and paths

Definition

- A *(smooth) curve* in a vector space V is a smooth map $\gamma\colon [a, b] \to V$.
- A *piecewise smooth curve* on V is a continuous map $\gamma\colon [a, b] \to V$ for which there is a collection of points $a = a_0 < a_1 < \cdots < a_k = b$ such that the restriction of γ to each interval $[a_i, a_{i+1}]$ is a smooth curve.

Curves and paths

Definition

- A topological space X is *connected* if whenever we have $X = A \cup B$ with A and B open and $A \cap B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$.
- A topological space X is *path-connected* if for any two points $p, q \in X$ there is a continuous path $\gamma\colon [a, b] \to X$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

Curves and paths

The truth: An open set in a vector space is connected if and only if it is smooth path-connected.

Curves and paths

Theorem

If a holomorphic function $f : U \to \mathbb{C}$ satisfies $f' \equiv 0$, then f is locally constant. If U is connected, then f is constant.

Curves and paths

Proof.

Let $p, q \in U$ be two points that can be connected by a path γ and consider the composition $\mathrm{Re} \circ f \circ \gamma \colon [a, b] \to \mathbb{R}$.
Taking the derivative we have

$$\mathrm{Re} \circ df|_{\gamma(t)} \gamma'(t) = 0 \qquad \text{for all } t.$$

Hence $\mathrm{Re} \circ f \circ \gamma \colon [a, b] \to \mathbb{R}$ is constant and the same applies to the imaginary part. Therefore $f(p) = f(q)$.
Since in an open set $U \subset C$ any point p has a connected neighbouhood we conclude that f is locally constant.
The argument also makes clear that f is constant on path-connected components of U. □

Curves and paths

Theorem

- *If* $f\colon U \to \mathbb{C}$ *is analytic and not constant and* U *is connected, then the zeros of* f *are isolated.*
- *If* $f, g\colon U \to \mathbb{C}$ *are analytic and the set* $S = \{z\colon f(z) = g(z)\}$ *has an acummulation point in* U*, then* $f = g$*.*

Curves and paths

Proof.

Clearly the first claim implies the second.
If f is not constant in a neighbouhood of z_0, then, by a Lemma from last lecture, there is an analytic isomorphism φ such that

$$f \circ \varphi(z - z_0) = a_n (z - z_0)^n,$$

showing that z_0 is the only point mapped to 0 in that open set.

Curves and paths

Proof.

Therefore, either $f \equiv 0$ in a neighbourhood of z_0 or z_0 is an isolated zero.
If $f \equiv 0$ in a neighbourhood of some point z_0, let

$$S = \{z \in U : f \equiv 0 \text{ in a neighbourhood of } z\}.$$

S is open.
The complement of S is also open by the previous argument. □

Integrals

Definition

Given $F\colon [a,b] \to \mathbb{C}$ with $F(t) = u(t) + iv(t)$ we define

$$\int_a^b F(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

Theorem (Fundamental Theorem of Calculus)

Given $F\colon [a,b] \to \mathbb{C}$ continuous, define

$$I\colon [a,b] \to \mathbb{C}, \qquad I(t) = \int_a^t F(\tau)d\tau.$$

Then $\frac{dI}{dt} = F$.

Integrals

Definition

Given $f\colon \mathbb{C} \to \mathbb{C}$ and $\gamma\colon [a,b] \to \mathbb{C}$, we define the *integral of f over γ* to be

$$\int_\gamma f := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

This does not depend on the parametrization of γ.

Integrals

Example

Take $f(z) = 1/z$ and $\gamma\colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$. Then

$$\int_\gamma f = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} = 2\pi i.$$

Integrals

Definition

A *primitive* of a continuous function $f\colon U \to \mathbb{C}$ is a holomorphic function $g\colon U \to \mathbb{C}$ such that $\frac{dg}{dz} = f$.

Theorem

Let g be a primitive of f and let $\gamma\colon [a,b] \to U$ be a curve. Then

$$\int_\gamma f = g(\gamma(b)) - g(\gamma(a)).$$

Integrals

Proof.

Consider $G\colon [a,b] \to \mathbb{C}$, given by $G(t) = g(\gamma(t))$. Then

$$\frac{dG}{dt} = g'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t).$$

Integrating we obtain

$$g(\gamma(b)) - g(\gamma(a)) = \int_a^b \frac{dG}{dt}dt = \int_a^b f(\gamma)\gamma' dt = \int_\gamma f.$$

□

Integrals

Example

Take $f(z) = z^k$ for $k \neq -1$ and $\gamma\colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$. Then $g(z) = \frac{z^{k+1}}{k+1}$ is a primitive of f, hence

$$\int_\gamma f = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} = g(e^{2\pi i}) - g(1) = 0.$$

Integrals

Theorem

Let $f\colon U \to \mathbb{C}$ *be a continuous function and assume that for every path* $\gamma\colon [a,b] \to U$ *with* $\gamma(a) = \gamma(b)$ *we have*

$$\int_\gamma f = 0.$$

Then f *has a primitive* $g\colon U \to \mathbb{C}$.

Integrals

Proof.

Fix $z_0 \in U$ and define

$$g(z) = \int_{z_0}^{z} f$$

By hypothesis, this integral is independent of the path chosen connecting z_0 to z.

We have

$$\frac{g(z+h) - g(z)}{h} = \frac{1}{h} \int_{z}^{z+h} f.$$

Integrals

Proof.

Define φ by the equatility $f(w) = f(z) + \varphi(w)$, so $\varphi(z) = 0$. Then

$$\begin{aligned} \frac{1}{h}\int_z^{z+h} f &= \frac{1}{h}\int_z^{z+h} f(z) + \varphi \\ &= f(z) + \frac{1}{h}\int_z^{z+h} \varphi \end{aligned}$$

But

$$\begin{aligned} \left| \frac{1}{h}\int_z^{z+h} \varphi \right| &< \frac{1}{h}\int_z^{z+h} |\varphi| \\ &< \max |\varphi||_{[z,z+h]} \\ &< \epsilon. \end{aligned}$$