Analysis in one complex variable
Lecture 4 – Integration II

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Last lecture: Integration I

Recall:

- We defined

$$\int_\gamma f(z)dz,$$

- If $\gamma\colon [a,b] \to U$ and $\frac{dg}{dz} = f$ on U, then

$$\int_\gamma f(z)dz = g(\gamma(b)) - g(\gamma(a))$$

- If $\int_\gamma f(z)dz = 0$ for every loop γ, then f has a primitive.

Last lecture: Integration I

Recall:

- $\int_{S^1} \frac{1}{z} = 2\pi i$
- $\int_{S^1} z^n = 0$ for $n \neq -1$

Primitives

Question of the day

Given $f: U \to \mathbb{C}$, when does it have a primitive, that is, where there is a holomorphic $g: U \to \mathbb{C}$ such that

$$\frac{dg}{dz} = f?$$

The answer depends on f and on U.

Homotopies

Definition

A *homotopy* between continuous functions $f_0, f_1 \colon X \to Y$, a continuous function $H \colon X \times [0,1] \to Y$ such that

$$H(x,0) = f_0(x) \qquad \text{and} \qquad H(x,1) = f_1(x).$$

Intuitively, a homotopy is a continuous way to deform one continuous function into another.

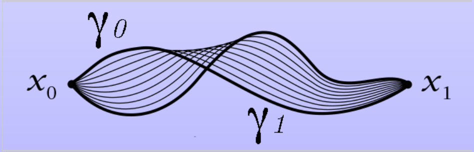
Homotopies of paths and loops

Definition

A *homotopy between paths* $\gamma_0, \gamma_1 \colon [0,1] \to X$, a continuous function $H \colon [0,1] \times [0,1] \to X$ such that

$$H(x,0) = \gamma_0(x) \qquad \text{and} \qquad H(x,1) = \gamma_1(x),$$

$$H(0,t) = \gamma_0(0) \qquad \text{and} \qquad H(1,t) = \gamma_0(1).$$



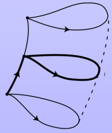
Homotopies of paths and loops

Definition

A *loop on X* is a map $\gamma\colon S^1 \to X$.
A *loop with base point on X* is a map $\gamma\colon ([0,1],\{0,1\}) \to (X, x_0)$.

Remark

For us a **homotopy between loops** *does not need to preserve any point of the loop, while a* **homotopy of loops with base points** *will keep the basepoint fixed.*



Homotopies of paths and loops

Definition

A connected subset $U \subset \mathbb{C}$ is *simply-connected* if every loop in U is homotopic to the constant loop.

Remark

In the course geometry and topology, we require this to be a homotopy of loops with base point. For open sets in vector spaces these two notions agree.

Homotopies of paths and loops

Definition

A subset $U \subset \mathbb{C}$ is *convex* if for all $z, w \in U$ the line connecting z and w is also in U, that is

$$tz + (1-t)w \in U \qquad \text{for all } t \in [0,1].$$

Lemma

Every convex set $U \subset \mathbb{C}$ is simply-connected.

Proof.

Given a loop $\gamma\colon S^1 \to U$ define

$$H\colon S^1 \times [0,1] \to U, \qquad H(x,t) = (1-t)\gamma(x) + t\gamma(1).$$

□

Homotopies of paths and loops

Definition

A subset $U \subset \mathbb{C}$ is *star-shaped* if there is $z \in U$ such that for all $w \in U$ the line connecting z and w is also in U.

Exercise

Every star-shaped $U \subset \mathbb{C}$ is simply-connected.

Primitives

Question of the day

Given $f : U \to \mathbb{C}$, when does it have a primitive?

Theorem

Let $f : U \to \mathbb{C}$ be a holomorphic function defined on a simply connected set U. Then f has a primitive in U.

Fun with logarithms

Example

The function

$$f\colon \mathbb{C}\backslash\{0\} \to \mathbb{C}, \qquad f(z) = \frac{1}{z}$$

is holomorphic but does not have a primitive.
If we restrict the domain of f to $\mathbb{C}\backslash\mathbb{R}_-$, it does!

Fun with logarithms

Example

Let g be a primitive of $f : \mathbb{C} \backslash \mathbb{R}_- \to \mathbb{C}$, given by $f(z) = 1/z$ with $g(1) = 0$
Since f is analytic, g is also analytic.
Over $\mathbb{R}_+$ we have $e^{g(z)} = z$, so this holds in all $\mathbb{C} \backslash \mathbb{R}_-$.
We write $g(z) = \log z$ and call g the *principal value* of the logarithm.
Keep in mind that $e^{2k\pi i} = 1$. How many g's are there with $e^g = \mathrm{Id}$?

Primitives

Theorem

Let $f : U \to \mathbb{C}$ be a holomorphic function defined on a simply connected set U. Then f has a primitive in U.

Primitives

Proof.

1. Prove a version when U is a rectangle:

$$g(z) = \int_{z_0}^{z} f.$$

2. **Show that the result depends only on the homotopy class of the path.**
3. Conclude that $\int_\gamma f(z)dz = 0$ for all closed loops.

□