Analysis in one complex variable
Lecture 4 – Integration II

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Primitives

Theorem

Let $f : U \to \mathbb{C}$ be a holomorphic function defined on a simply connected set U. Then f has a primitive in U.

Primitives

Proof.

1. **Prove a version when U is a rectangle:**

$$g(z) = \int_{z_0}^{z} f.$$

2. Show that the result depends only on the homotopy class of the path.
3. Conclude that $\int_\gamma f(z)dz = 0$ for all closed loops.

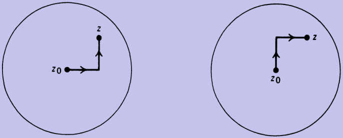
Step 1

Claim

Given a holomorphic function ,f, the function

$$g(z) = \int_{z_0}^{z} f.$$

where the integral is taken over a path made of two segments: one horizontal and one vertical



Does not depend on the path chosen.

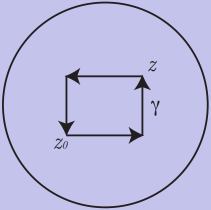
Step 1

Proof.

Equivalently we need to show that

$$\int_{\gamma} f(z)dz = 0$$

if γ is the boundary of a rectangle



Step 1

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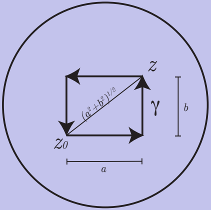
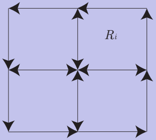


Step 1

Proof.

We compute the integral by dividing R into four equal pieces



and taking the one with the biggest integral (in absolute value) we have

$$\left|\int_{\partial R} f(z)dz\right| \leq 4\left|\int_{\partial R_i} f(z)dz\right|$$

Step 1

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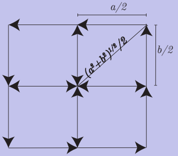


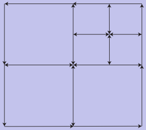
and taking the one with the biggest integral (in absolute value) we have

$$\left| \int_{\partial R} f(z)dz \right| \leq 4 \left| \int_{\partial R_i} f(z)dz \right|$$

Step 1

Repeat the same with R_i

Proof.

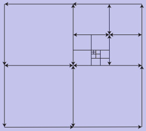


$$\left| \int_{\partial R} f(z)dz \right| \leq 4^2 \left| \int_{\partial R_{ij}} f(z)dz \right|$$

Step 1

After n divisions,

Proof.



$$\left| \int_{\partial R} f(z)dz \right| \leq 4^n \left| \int_{\partial R_{i_i \ldots i_n}} f(z)dz \right|$$

Step 1

Proof.

Consider the sequence (z_n), where z_n is the center of the rectangle $R_{i_i \ldots i_n}$. This sequence converges, say, to $\tilde{z}$.
Since each $R_{i_i \ldots i_n}$ is closed $\tilde{z} \in R_{i_i \ldots i_n}$ for all n.
Since f is holomorphic, we can approximate f in a neighbourhood of $\tilde{z}$ as

$$f(z) = f(\tilde{z}) + f'(\tilde{z})(z - \tilde{z}) + (z - \tilde{z})h(z),$$

where $\lim_{z \to \tilde{z}} |h(z)| = 0$.
In particular, given $\epsilon > 0$, for all n large enough $\sup\{|h(z)| : z \in R_{i_1 \ldots i_n}\} < \epsilon$.

Step 1

Proof.

Now we compute

$$\left|\int_{\partial R} f(z)dz\right| \leq 4^n \left|\int_{\partial R_{i_1\ldots i_n}} f(z)dz\right|$$
$$= 4^n \left|\int_{\partial R_{i_1\ldots i_n}} (f(\tilde{z}) + f'(\tilde{z})(z-\tilde{z}) + (z-\tilde{z})h(z))dz\right|.$$

The first two terms vanish because the integrand is holomorphic.

Step 1

Proof.

$$\begin{aligned}\left|\int_{\partial R} f(z)dz\right| &\leq 4^n \left|\int_{\partial R_{i_1\dots i_n}} (z-\tilde{z})h(z)dz\right| \\ &\leq 4^n \int_{\partial R_{i_1\dots i_n}} \sup(|z-\tilde{z}|)\sup(|h(z)|)|dz| \\ &\leq 4^n \frac{\sqrt{a^2+b^2}}{2^n}\epsilon\; 2\frac{a+b}{2^n} \\ &= 2(a+b)\sqrt{a^2+b^2}\;\epsilon\end{aligned}$$

which can be made as small as one wants. □

Step 1

Conclusion

The function

$$g(z) = \int_{z_0}^{z} f.$$

where the integral is taken over a path made of two segments: one horizontal and one vertical is well defined.

Step 1

Lemma

Given a holomorphic function, f, the function

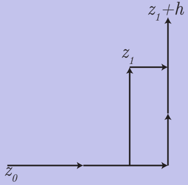
$$g(z) = \int_{z_0}^{z} f.$$

as defined above is holomorphic and $g' = f$.

Step 1

Proof.

Fix a point z_1 and compute $g(z_1 + h) - g(z_1)$.



Step 1

Proof.

Since the integral of f over boundary of rectangles is zero, we have

$$
\begin{aligned}
g(z_1 + h) &= \int_{z_0}^{z_1+h} f(z)dz \\
&= \int_{z_0}^{z_1} f(z)dz + \int_{z_1}^{z_1+h} f(z)dz \\
&= g(z_1) + \int_{z_1}^{z_1+h} f(z)dz.
\end{aligned}
$$

That is

$$g(z_1 + h) - g(z_1) = \int_{z_1}^{z_1+h} f(z)dz.$$

Step 1

Proof.

Define $f(z) = f(z_1) + \psi(z)$, so $\lim_{z \to z_1} \psi(z) = 0$. Then

$$\int_{z_1}^{z_1+h} f(z)dz = \int_{z_1}^{z_1+h} (f(z_1) + \psi(z))dz$$
$$= f(z_1)h + \int_{z_1}^{z_1+h} \psi(z)dz.$$

Step 1

Proof.

Hence

$$\left|\frac{g(z_1+h)-g(z_1)}{h}-f(z_1)\right| = \frac{1}{h}\left|\int_{z_1}^{z_1+h}\psi(z)dz\right|$$
$$\leq \frac{1}{h}\int_{z_1}^{z_1+h}|\psi(z)||dz|$$
$$\leq \frac{1}{h}\sup(\psi|_{B_h(z_1)})|h|$$

□

Step 1

Corollary (Holomorphic Poincaré Lemma in 1 dimension)

Any holomorphic function defined on the disc has a primitive.