Analysis in one complex variable
Lecture 5 – Local Cauchy Theorem

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Recall

- $\int_{S^1} \frac{dz}{z} = 2\pi i$
- If $f : U \to \mathbb{C}$ is holomorphic and $\gamma_0, \gamma_1 : S^1 \to U$ are homotopic, then

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$$

Cauchy integral formula

Theorem

Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open set. Let $z_0 \in U$ and D be a disc with

$$z_0 \subset D \subset \bar{D} \subset U.$$

Let $\gamma : [0,1] \to U$ be the boundary of D traced counterclockwise. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Cauchy integral formula

Proof.

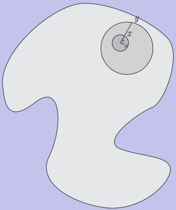
- Topological: homotope γ to the boundary of a smaller disc, D_r, centred at z_0,
- Analytical: estimate and integral over ∂D_r,
- Easy: compute a simpler integral.

Cauchy integral formula

Proof.

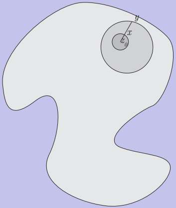
Homotopy

Homotope γ to the boundary of a smaller disc, D_r, centred at z_0
For r small enough, the disc of radius r centred at z_0, D_r is inside D.



Cauchy integral formula

Proof.



It is visually clear that ∂D_r and ∂D are homotopic within $U \backslash \{z_0\}$.
The homotopy is obtain by picking the rays from z_0 and seeing where they meet ∂D_r and ∂D.

Cauchy integral formula

Proof.

The function $g(z) = \frac{f(z)-f(z_0)}{z-z_0}$ is holomorphic on $U \backslash \{z_0\}$. By homotopy invariance

$$\int_{\partial D} g\, dz = \int_{\partial D_r} g\, dz.$$

Cauchy integral formula

Proof.

The estimate

Since f is holomorphic, g is bounded on D, say $|g(z)| \leq B$. Then

$$0 \leq \left| \int_{\partial D} g \, dz \right| = \left| \int_{\partial D_r} g \, dz \right| \leq \int_{\partial D_r} |g| \, |dz| \leq B 2\pi r.$$

Since $r > 0$ is arbitrary (small) we conclude that

$$\int_{\partial D} g \, dz = 0.$$

Cauchy integral formula

Proof.

Simple integral

$$0 = \int_{\partial D_r} \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$\int_{\partial D} \frac{f(z)}{z - z_0} dz = \int_{\partial D_r} \frac{f(z)}{z - z_0} dz = \int_{\partial D_r} \frac{f(z_0)}{z - z_0} dz = 2\pi i f(z_0).$$

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