Analysis in one complex variable
Lecture 5 – Local Cauchy Theorem

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Recall

Theorem (Cauchy Integral Formula)

Let $f: U \to \mathbb{C}$ be a holomorphic function defined on an open set. Let $z_0 \in U$ and D be a disc with

$$z_0 \subset D \subset \bar{D} \subset U.$$

Let $\gamma: [0, 1] \to U$ be the boundary of D traced counterclockwise. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Cauchy integral formula

Theorem

Every holomorphic function on $U \subset \mathbb{C}$ is analytic on U.

Cauchy integral formula

Theorem

Let f be holomorphic on a disc $\bar{D}_R(z_0)$. Then f has a power series expansion

$$f(z) = \sum a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{n!} \frac{d^n f}{dz^n}(z_0) = \frac{1}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

Further, if $B = \sup\{|f(z)| : z \in \partial D_R(z_0)\}$, then

$$a_n \leq \frac{B}{R^n}$$

hence the radius of convergence is at least R.

Cauchy integral formula

Proof.

Start with a remark that has no bearing on the proof. For $z \in D_R(z_0)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(\xi)}{\xi - z} d\xi$$

Taking the derivative, we get

$$\frac{df}{dz}(z) = \frac{1}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(\xi)}{(\xi - z)^2} d\xi$$

$$\frac{d^2 f}{dz^2}(z) = \frac{2}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(\xi)}{(\xi - z)^3} d\xi$$

Cauchy integral formula

Proof.

$$\frac{d^n f}{dz^n}(z) = \frac{n!}{2\pi i}\int_{\partial D_R(z_0)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

In particular f is infinitely many times differentiable and

$$a_n = \frac{1}{n!}\frac{d^n f}{dz^n}(z_0) = \frac{1}{2\pi i}\int_{\partial D_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

Cauchy integral formula

Proof.

$$
\begin{aligned}
|a_n| &= \left| \frac{1}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right| \\
&\leq \frac{1}{2\pi} \int_{\partial D_R(z_0)} \frac{|f(\xi)|}{R^{n+1}} |d\xi| \\
&\leq \frac{1}{2\pi} \frac{B}{R^{n+1}} 2\pi R \\
&= \frac{B}{R^n}
\end{aligned}
$$

Cauchy integral formula

Proof.

Conclusion: the series $\sum a_n z^n$ is analytic and has radius of convergence at least R.
But does the series converge to f?

Cauchy integral formula

Proof.

For that we use the series $\frac{1}{1-x} = 1 + x + x^2 + \ldots$:

$$\begin{aligned}
\frac{1}{\xi - z} &= \frac{1}{\xi - z_0 - (z - z_0)} \\
&= \frac{1}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \\
&= \frac{1}{\xi - z_0} \left(1 + \frac{z - z_0}{\xi - z_0} + \left(\frac{z - z_0}{\xi - z_0} \right)^2 + \ldots \right) \\
&= \left(\frac{1}{\xi - z_0} + \frac{z - z_0}{(\xi - z_0)^2} + \frac{(z - z_0)^2}{(\xi - z_0)^3} + \ldots \right)
\end{aligned}$$

converges for $|z - z_0| < |\xi - z_0|$.

Cauchy integral formula

Proof.

Therefore

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial D_R(z_0)} \frac{f(\xi)}{\xi - z}d\xi \\ &= \frac{1}{2\pi i}\int_{\partial D_R(z_0)} f(\xi)\left(\frac{1}{\xi - z_0} + \frac{(z-z_0)}{(\xi - z_0)^2} + \frac{(z-z_0)^2}{(\xi - z_0)^3} + \dots\right)d\xi \\ &= \sum a_n (z-z_0)^n. \end{aligned}$$

□

Cauchy integral formula

Corollary (Liouville's Theorem)

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, then f is constant.

Proof.

Say $|f| \leq B$ for some B. Given z and any $R > 0$ we have

$$\begin{aligned} |f'(z)| &= \frac{1}{2\pi} \left| \int_{\partial B_R(z)} \frac{f(\xi)}{(\xi - z)^2} d\xi \right| \\ &\leq \frac{1}{2\pi} \int_{\partial B_R(z)} \frac{|f(\xi)|}{R^2} |d\xi| \\ &\leq \frac{1}{2\pi} \frac{B}{R^2} 2\pi R = \frac{B}{R} \end{aligned}$$

So $f'(z) = 0$ and f is constant. □

Cauchy integral formula

Corollary

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and there is $n \in \mathbb{N}$ such that $\frac{f(z)}{(|z|+1)^n}$ is bounded, then f is a polynomial of degree at most n.

Proof.

Show that $\frac{d^{n+1}f}{dz^{n+1}}$ vanishes. $\square$

Cauchy integral formula

Corollary (Morera's Theorem)

Let $f : U \to C$ be continuous and assume that $\int_{\partial R} f dz = 0$ for all rectangles $R \subset U$. Then f is holomorphic.

Proof.

Being holomorphic is a local condition, so it is enough to show that f is holomorphic in a neighbouhood of every point in U. Let $z \in U$ and $D \subset U$ be a disc containing z. Since $\int_{\partial R} f dz = 0$ for all rectangles $R \subset U$, f has primitive, g, defined in D (D is simply-connected):

$$\frac{dg}{dz} = f.$$

Since g is holomorphic, it is analytic and hence so is f and therefore f is holomorphic on D. □