Analysis in one complex variable
Lecture 5 – The problem

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May 2020
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Recall

- Given a group G, the commutator subgroup of G, $[G, G]$ is generated by elements of the form

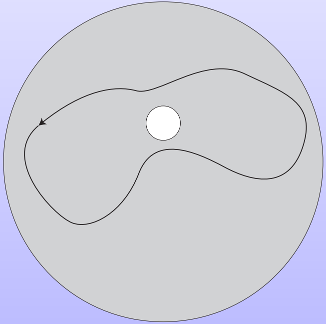
$$ghg^{-1}h^{-1}.$$

  $[G, G] \lhd G$ is a normal subgroup and $G/[G, G]$ is Abelian.
- If $f\colon U \to \mathbb{C}$ is holomorphic and $\gamma_0, \gamma_1\colon [0, 1] \to \mathbb{C}$ are two loops which are homotopic, then

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz.$$

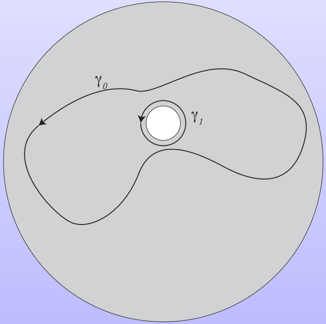
The problem

$$\int_\gamma f dz = ?$$



The problem

$$\int_\gamma f dz = ?$$



The problem

For example, take

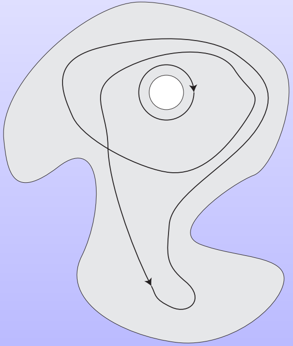
$$f\colon \mathbb{C}\backslash\{0\} \to \mathbb{C}, \qquad f(z) = 1/z$$

and

$$\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$$
$$\gamma(t) = 3\sin(2\pi t) + i(4\cos(2\pi t) + \sin(4\pi t))$$

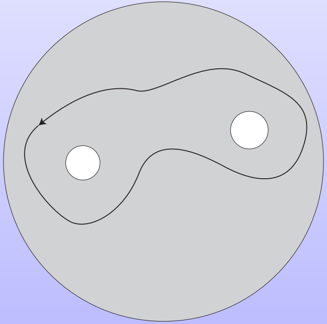
The problem

$$\int_\gamma f dz = ?$$



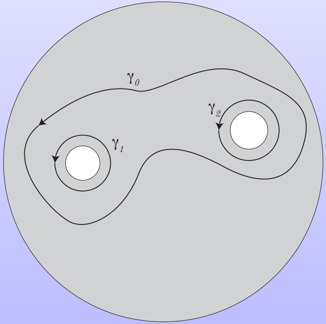
The problem

$$\int_\gamma f dz = ?$$



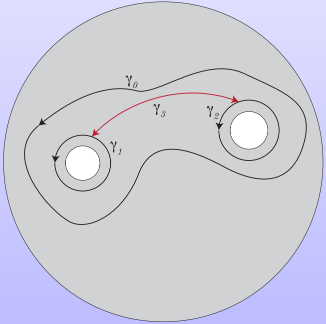
The problem

$$\int_\gamma f dz = ?$$



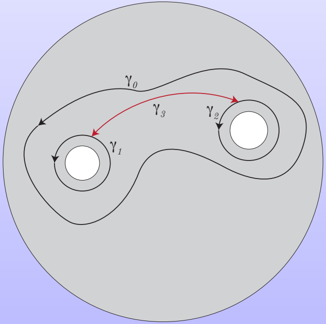
The problem

$$\int_\gamma f dz = ?$$



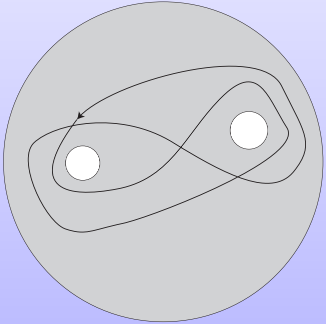
The problem

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz + \int_{\gamma_3} f dz + \int_{\gamma_1} f dz - \int_{\gamma_3} f dz$$



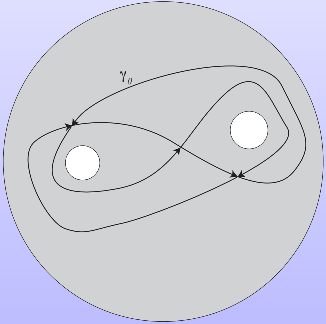
The problem

$$\int_\gamma f dz = ?$$



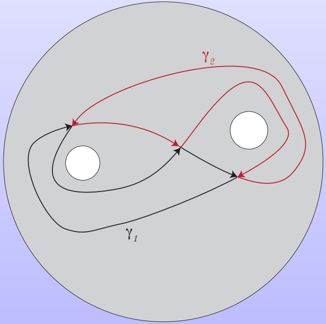
The problem

$$\int_\gamma f dz = ?$$



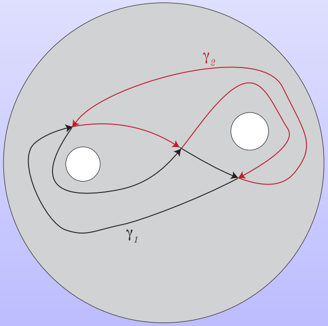
The problem

$$\int_{\gamma} f dz = ?$$



The problem

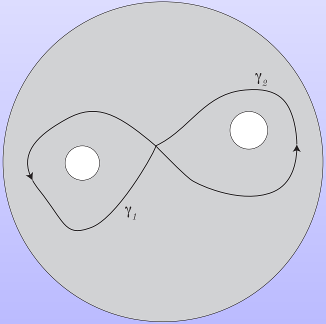
$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz$$



The problem

$$\int_{\gamma_0} f dz = ?$$

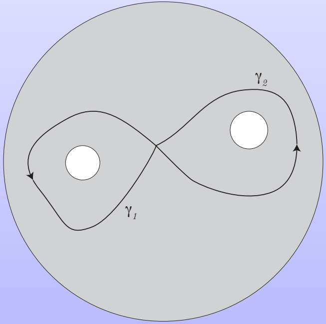
where $\gamma_0 = \gamma_1 * \gamma_2 * \gamma_1^{-1} * \gamma_2^{-1}$



The problem

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz - \int_{\gamma_1} f dz - \int_{\gamma_2} f dz = 0$$

where $\gamma_0 = \gamma_1 * \gamma_2 * \gamma_1^{-1} * \gamma_2^{-1}$



The problem

- The space of base-point loops in U up to homotopy is a group with concatenation of loops as group operation.
- This group is called the fundamental group of U and denoted by $\pi_1(U; x_0)$.
- $\pi_1(U; x_0)$ is often not Abelian.

The problem

- Integration of a holomorphic function gives a group homormophism:

$$\int f\,dz\colon \pi_1(U;x_0) \to \mathbb{C}, \qquad \gamma \mapsto \int_\gamma f dz.$$

- Since $\mathbb{C}$ is Abelian, any commutator $\gamma_1 * \gamma_2 * \gamma_1^{-1} * \gamma_2^{-1}$ is mapped to 0.

$$\int f\,dz\colon \pi_1(U;x_0)/[\pi_1(U;x_0), \pi_1(U;x_0)] \to \mathbb{C}$$

The problem

But $\pi_1(U; x_0)/[\pi_1(U; x_0), \pi_1(U; x_0) = H_1(U)$, so in fact we have

Theorem (Global Cauchy Theorem)

Given a holomorphic function $f\colon U \to \mathbb{C}$, integration induces a map in homology

$$\int f\,dz\colon H_1(U) \to \mathbb{C}, \gamma \mapsto \int_\gamma f dz,$$

that is, if γ_0 and γ_1 are homologous, then

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f dz.$$