Analysis in one complex variable
Lecture 6 – Global Cauchy

Gil Cavalcanti

Utrecht University

May 2020
Utrecht

Recall

Claim

If two loops $\gamma_i \colon [0,1] \to U$, $i = 0, 1$ *are homotopic and* $f \colon U \to \mathbb{C}$ *is holomorphic, then*

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz.$$

Quest

Get away from homotopies and into a more general setting.

Winding number

Definition

Given a loop $\gamma\colon [0,1] \to \mathbb{C}$ and a point $\alpha \notin \gamma([0,1])$, the *winding number of* γ *with respect to* α is

$$W(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - \alpha}.$$

Winding number

Example

Let γ be the unit circle traced counterclowise n times, that is

$$\gamma\colon [0,1] \to \mathbb{C}, \qquad \gamma(t) = e^{2\pi i n t}.$$

The winding number of γ with respect to the origin is

$$\begin{aligned} W(\gamma, 0) &= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} \\ &= \frac{1}{2\pi i} \int_0^1 \frac{\gamma'}{\gamma} dt \\ &= \frac{1}{2\pi i} \int_0^1 \frac{2\pi i n\, e^{2\pi i n t}}{e^{2\pi i n t}} dt \\ &= \int_0^1 n dt \end{aligned}$$

Winding number

Example

Let γ be the unit circle traced counterclowise n times, that is

$$\gamma\colon [0,1] \to \mathbb{C}, \qquad \gamma(t) = e^{2\pi i n t}.$$

The winding number of γ with respect to 2 is

$$W(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-2} = 0,$$

by homotopy invariance of the integral.

Winding number

Lemma

The winding number is an integer.

Proof.

Define

$$F\colon [0,1] \to \mathbb{C}, \qquad F(t) = \int_0^t \frac{\gamma'(t)}{\gamma(t)-\alpha}dt$$

So $F(1) = 2\pi i W(\gamma, \alpha)$.

Winding number

Proof.

$$F\colon [0,1] \to \mathbb{C}, \qquad F(t) = \int_0^t \frac{\gamma'(t)}{\gamma(t) - \alpha} dt$$

Consider the path $(\gamma(t) - \alpha)e^{-F(t)}$.
We have

$$\frac{d}{dt}((\gamma - \alpha)e^{-F}) = (\gamma' - (\gamma - \alpha)\frac{\gamma'}{\gamma - \alpha})e^{-F} = 0$$

Winding number

Proof.

Therefore $(\gamma - \alpha)e^{-F} = C$ for some constant C. Since $\gamma(0) = \gamma(1)$, we have

$$Ce^{F(0)} = (\gamma(0) - \alpha) = (\gamma(1) - \alpha) = Ce^{F(1)}$$

Hence

$$e^{F(1)-F(0)} = 1 \Rightarrow F(1) - F(0) = 2\pi i n,$$

for some $n \in \mathbb{Z}$.
Then

$$W(\gamma, \alpha) = \frac{1}{2\pi i} F(1) = n.$$

□

Winding number

Lemma

For any given loop γ, winding number is a continuous function on α.

Proof.

Follows from continuous dependence of integrals on parameters

$$W(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - \alpha}.$$

$\square$

Winding number

Lemma

For any given loop γ, winding number is constant on connected components of $\mathbb{C}\backslash\gamma([0,1])$.

Lemma

For any given loop γ, if $\alpha \in \mathbb{C}\backslash\gamma([0,1])$ is a point in an unbounded component of $\mathbb{C}\backslash\gamma([0,1])$ then $W(\gamma,\alpha) = 0$.

Proof.

Just look at

$$W(\gamma,\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\alpha}.$$

□

Winding number

Lemma

For any given point, α, $W(\gamma, \alpha)$ does not change under sufficiently small deformations of γ.

Proof.

Just look at

$$W(\gamma, \alpha) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - \alpha}.$$

□

Chains

Definition

- The *space of 1-chains* in U, $C_1(U;\mathbb{C})$ is the vector space generated by all the paths in U.
- A *1-chain* is an element of $C_1(U;\mathbb{C})$, that is, it is a finite formal combination of paths:

$$\sigma = \sum_{i=1}^{n} \lambda_i \gamma_i, \qquad \lambda_i \in \mathbb{C}.$$

Chains

- Extend integration to 1-chains by linearity: if $\sigma = \sum \lambda_i \gamma_i$

$$\int_\sigma f\,dz = \sum_i \lambda_i \int_{\gamma_i} f\,dz.$$

Chains

Definition

- The *space of 0-chains* in U, $C_0(U;\mathbb{C})$ is the vector space generated by all the points in U.
- A *0-chain* is an element of $C_0(U;\mathbb{C})$, that is, it is a finite formal combination of points:

$$\sigma = \sum_{i=1}^{n} \lambda_i p_i, \qquad \lambda_i \in \mathbb{C}.$$

Chains

Definition

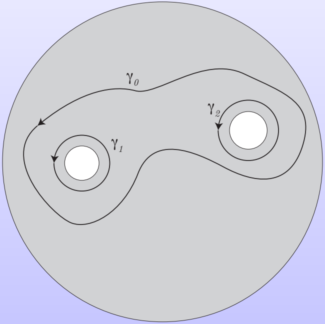
The *boundary* is a linear operation $\partial\colon C_1(U;\mathbb{C}) \to C_0(U;\mathbb{C})$ defined by

$$\partial\gamma = \gamma(1) - \gamma(0).$$

Definition

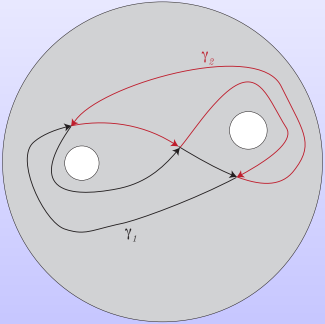
A 1-chain σ is a *cycle* if $\partial\sigma = 0$.

The problem



Want: $\gamma_0 = \gamma_1 + \gamma_2$

The problem



Want: $\gamma_0 = \gamma_1 + \gamma_2$

Homology

Definition

- Given a set $U \subset \mathbb{C}$, two 1-chains $\sigma_0, \sigma_1 \colon [0,1] \to U$ are *homologous* if for every $\alpha \in \mathbb{C} \backslash U$

$$W(\sigma_0, \alpha) = W(\sigma_1, \alpha).$$

- A chain σ is *null homologous* if for all $\alpha \in \mathbb{C} \backslash U$

$$W(\sigma, \alpha) = 0.$$

Homology

Theorem

Given two loops $\gamma_0, \gamma_1 \colon [0,1] \to U \subset \mathbb{C}$

- *If* γ_0 *and* γ_1 *are homotopic, then they are homologous.*
- *If* γ_0 *and* γ_1 *are close together, then they are homologous.*

Homology

Theorem

- *Let U be an open set and σ a closed chain which is null homologous in U. Let $z_1, \ldots, z_n$ be a finite number of distinct points of U.*
- *Let γ_i $(i = 1, \ldots, n)$ be the boundary of a closed disc V_i contained in U, containing z_i and oriented counterclockwise.*
- *Assume that V_i does not intersect V_j if $i \neq j$.*
- *Let $m_i = W(\sigma, z_i)$.*
- *Let U^* be the set obtained by deleting $z_1 \ldots, z_n$ from U.*

Then σ is homologous to $\sum m_i \gamma_i$ in U^*.

Homology

Example

The homology class of a chain σ on $\mathbb{C}\backslash\{z_1, \ldots, z_k\}$ is determined by the winding numbers of σ around each of the points $\{z_1, \ldots, z_k\}$.

Homology

Proof.

Compute the winding numbers of both σ and $\sum m_i \gamma_i$ for points $\alpha \in \mathbb{C} \backslash U^*$.

- for $\alpha \notin U$, both winding numbers vanish.
- for $\alpha \in \{z_1, \ldots, z_n\}$, both winding numbers agree.

$\square$