Analysis in one complex variable
Lecture 6 – Global Cauchy

Gil Cavalcanti

Utrecht University

May 2020
Utrecht

Cauchy's Theorem

Theorem (Cauchy)

If two cycles γ_0 and γ_1 on U are homologous and $f : U \to \mathbb{C}$ is holomorphic, then

$$\int_{\gamma_0} fdz = \int_{\gamma_1} fdz.$$

Theorem (Cauchy equivalent)

Let γ be a null homologous cycle on U and let $f : U \to \mathbb{C}$ be holomorphic. Then

$$\int_{\gamma} f\, dz = 0.$$

Global Cauchy Integral Formula

Theorem (Global Cauchy Integral Formula)

Let γ be a null homologous cycle on $U \subset \mathbb{C}$ and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $z_0 \in U$ be a point which is not in γ, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz = W(\gamma, z_0) f(z_0).$$

Global Cauchy Integral Formula

Proof.

Since $\frac{f(z)}{z-z_0}$ is only defined in $U\backslash\{z_0\}$ we should not consider γ as a path on U but on $U^* = U\backslash\{z_0\}$.
From last lecture, we have that γ is homologous to $W(\gamma, z_0)\gamma_0$ on U^*, where γ_0 is a small circle around z_0 traced counterclockwise.
By Cauchy's theorem

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz = W(\gamma, z_0)\frac{1}{2\pi i}\int_{\gamma_0} \frac{f(z)}{z-z_0}dz$$
$$= W(\gamma, z_0)f(z_0)$$

□

Cauchy's Theorem

Idea of the proof.

1. subdividing a path that is part of cycle does not change the homology class of the cycle.
2. any path is homotopic to a rectangular path: a finite concatenation of horizontal and vertical paths.
3. if a rectangular cycle, γ is null homologous, there are finitely many rectangles, R_i, and complex numbers, m_i, such that $\gamma = \sum_{i=1}^{n} m_i \partial R_i$.

Cauchy's Theorem

Idea of the proof.

1. Subdividing a path that is part of cycle does not change the homology class of the cycle.

The winding number is given by

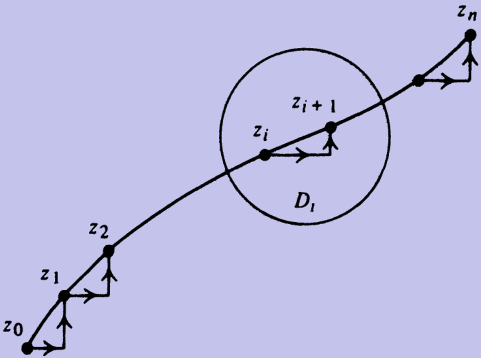
$$\int_\gamma \frac{1}{z-\alpha} dz$$

And integrals can be computed by splitting the domain in two and adding the result.

Cauchy's Theorem

Idea of the proof.

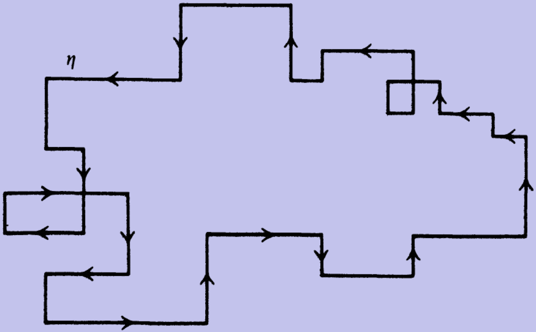
2. Any path is homotopic to a rectangular path: a finite concatenation of horizontal and vertical paths.



Cauchy's Theorem

Idea of the proof.

2. Any path is homotopic to a rectangular path: a finite concatenation of horizontal and vertical paths.

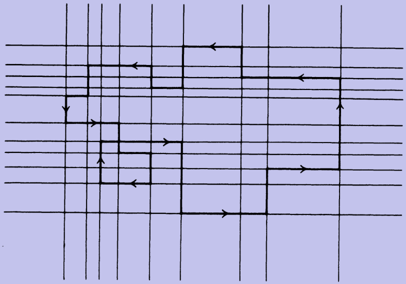


Cauchy's Theorem

Idea of the proof.

3. If a rectangular cycle, γ is null homologous, there are finitely many rectangles, R_i, and complex numbers, m_i, such that $\gamma = \sum_{i=1}^{n} m_i \partial R_i$.

Use the vertical and horizontal part of the cycle to split the plane into a finite collection of rectangles plus infinite strips.



Cauchy's Theorem

Idea of the proof.

- Pick $z_i \in R_i$ and let $m_i = W(\gamma, z_i)$.
- Only finitely many m_i are nonzero.
- Check that $\gamma = \sum_{i=1}^{n} m_i \partial R_i$.

Cauchy's Theorem

Idea of the proof.

$$\begin{aligned}\int_\gamma f\,dz &= \int_{\sum_{i=1}^n m_i \partial R_i} f\,dz.\\ &= \sum m_i \int_{\partial R_i} f\,dz.\\ &= 0\end{aligned}$$

□