Analysis in one complex variable
Lecture 7 – Logarithm

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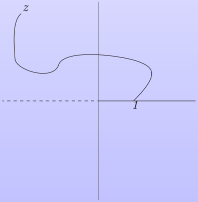
Logarithms

- Using existence of primitives on simply connected sets we defined

$$\log z = \int_0^z \frac{1}{\xi} d\xi$$

where the integral is over any path γ that does not cross the negative real numbers.



- Basic property:

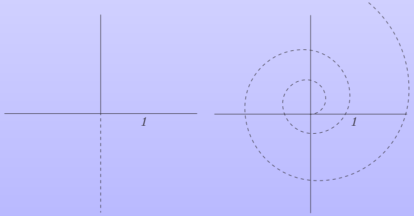
$$e^{\log z} = z.$$

Logarithms

- For $n \in \mathbb{Z}$, adding $2\pi i n$ to log will still give

$$e^{\log z + 2\pi i n} = z.$$

  The *principal value* of log is characterised by $\log 1 = 0$.
- Instead of removing the negative real numbers, we could have removed any path from 0 whose complement is simply connected.



Logarithms

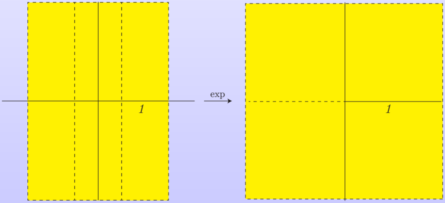
Basic facts

- For $\theta \in (-\pi, \pi)$, $\log(re^{i\theta}) = \log r + i\theta$,
- $\log(zw) = \log z + \log w \mod 2\pi i$,
- $\log e^z = z \mod 2\pi i$.

Logarithms

The open sets $U_0 = \mathbb{C} \backslash \mathbb{R}_-$ and $U_1 = \mathbb{C} \backslash \mathbb{R}_+$ have the properties

- $\exp^{-1}(U_i) = \cup_j V_{ij}$ is a disjoint union of open sets.



- $\exp\colon V_{ij} \to U_i$ is a diffeomorphism for all j.
- $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is a (smooth) covering space.

Logarithms

Given $f\colon U \to \mathbb{C}\backslash\{0\}$, can we define $\log f$?

Yes, if $f(U) \subset \mathbb{C}\backslash\mathbb{R}_-$, for example.

Proposition

If U is simply connected and f is continuous, there is a function $\log f\colon U \to \mathbb{C}$ such that

$$e^{\log f} = f.$$

Any other function $g\colon U \to \mathbb{C}$ with this property satisfies

$$g = \log f \mod 2\pi i$$

Logarithms

Proof.

Particular case 1: f is holomorphic.

Since $f \neq 0$, the function $\frac{f'}{f}$ is holomorphic in U. Since U is simply connected, $\frac{f'}{f}$ has a primitive:

$$\log f(z) := \log f(z_0) + \int_{z_0}^{z} \frac{f'}{f} dz.$$

Consider the function

$$F\colon U \to \mathbb{C}, \qquad F(z) = \frac{e^{\log f(z)}}{f(z)}$$

Notice

$$F(z_0) = \frac{e^{\log f(z_0)}}{f(z_0)} = 1$$

Logarithms

Proof.

$$F\colon U \to \mathbb{C}, \qquad F(z) = \frac{e^{\log f(z)}}{f(z)}$$

Then

$$\frac{dF}{dz} = \frac{f(z)\frac{f'(z)}{f(z)}e^{\log f(z)} - e^{\log f(z)}f'(z)}{f(z)}$$
$$= 0$$

Hence $F(z) = F(z_0) = 1$. (recall $F(z) = \frac{e^{\log f(z)}}{f(z)}$)
And $F(z) = 1 \Rightarrow e^{\log f(z)} = f(z)$.

Logarithms

Proof.

Particular case 2: U is an interval and f is differentiable.
Then $\frac{f'}{f}$ has a primitive (both real and imaginary parts of $\frac{f'}{f}$ have primitives).
Then repeat the previous argument.

Logarithms

Example

Given $\gamma\colon [0,1] \to \mathbb{C}$, $\gamma(t) = e^{2\pi i n t}$, since $[0,1]$ is simply connected, $\log \gamma$ exists.

Logarithms

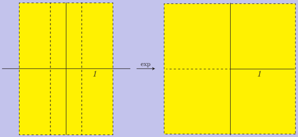
Proof.

Particular case 3: f is differentiable.
Then $\frac{df}{f}$ satisfies $d\frac{df}{f} = \frac{(d^2f)f - df \wedge df}{f^2} = 0$, that is it is closed.
Since U is simply connected, it has a primitive.
Then repeat the previous argument.

Logarithms

Proof.

General case: Pick a point $x_0 \in U$ and choose an arbitrary inverse $z_0 = \log(f(x_0))$.
If $f(x_0) \in U_0$ then x_0 has a neighbouhood $\tilde{U}$ that maps to U_0.



Let V_{0j} be the open set on $\mathbb{C}$ containing z_0 that maps diffeomorphically to U_0, then define

$$\log f(z) = \exp^{-1} f(z), \qquad z \in \tilde{U}.$$

Repeat this changing from U_0 to U_1 as needed.

Logarithms

Proof (The grown up version).

Want to find a lift $\tilde{f}$

$$\begin{array}{ccc} & & \mathbb{C} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \exp \\ U & \xrightarrow{f} & \mathbb{C}\backslash\{0\} \end{array}$$

There is one if and only if $f_*\pi_1(U) \subset \exp_* \pi_1(\mathbb{C}) = \{0\}$. □