Analysis in one complex variable
Lecture 7 – Sequences

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Recall

Theorem (Cauchy integral formula)

Let $f : U \to \mathbb{C}$ be holomorphic, let $z_0 \in U$ and let $D \subset U$ be a closed disc in U which contains z_0 in its interior, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z_0} d\xi.$$

Sequences

Theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on an open set U. Assume that for each compact subset $K \subset U$ the sequence converges uniformly on K, and let $f = \lim_{n\to\infty} f_n$. Then f is holomorphic.

Proof.

Let $z \in U$ and let $D \subset U$ be disc with z in its interior. Then

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\xi)}{\xi - z} d\xi$$

Taking the limit we get

Sequences

Proof.

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi \qquad \forall z \in U.$$

Since f is continuous on D, we can compute its derivative by derivating inside the integral.
Hence f is holomorphic. $\square$

Sequences

Theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on an open set U. Assume that for each compact subset $K \subset U$ the sequence converges uniformly on K, and let the limit function be f. Then $(f_n')_{n\in\mathbb{N}}$ converges uniformily on compact sets to f'.

Proof.

As before

$$f_n'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f_n(\xi)}{(\xi - z)^2} d\xi$$

Taking the limit we obtain

Sequences

Proof.

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{(\xi - z)^2} d\xi = f'(z).$$

□

Sequences

Example (Riemann zeta function)

Let $\zeta(z) = \sum_{n\in\mathbb{N}} \frac{1}{n^z}$, with $\mathrm{Re}(z) > 1$.
Notice that the summand $n^{-z} = e^{-z\log n}$, is holomorphic, hence so are the partial sums.
For $z = x + iy$,

$$|e^{-z\log n}| = |e^{-x\log n}||e^{iy\log n}| = |e^{-x\log n}| = \frac{1}{n^x},$$

which converges absolutely for $x > c > 1$.