Analysis in one complex variable
Lecture 7 – Laurent Series

Gil Cavalcanti

Utrecht University

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Utrecht

Recall

- If $f = \sum_{n=0}^{\infty} a_n z^n$ is analytic, its radius of convergence satisfies $\frac{1}{r} = \limsup |a_n|^{1/n}$.
- If f is holomorphic, then it is analytic,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

If D is centered at z_0 and has radius r, then the radius of convergence of the series is at least r.

Laurent series

- If $f = \sum_{n=0}^{\infty} a_n z^n$ is analytic, with radius of convergence r, then $g(z) := f(1/z) = \sum_{n=0}^{\infty} a_n z^{-n}$ converges for $|z| > 1/r$.
- Given a series $f = \sum_{n=-\infty}^{\infty} a_n z^n$, we let

$$f_+ = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad f_- = \sum_{n=-1}^{-\infty} a_n z^n$$

Laurent series

Lemma

If the radius of convergence of f_+ is R and of $f_-(1/z)$ is $1/r$ then f is analytic on the annulus $A = \{z \in \mathbb{C} \colon r < |z| < R\}$.

Proof.

For each $z_0 \in A$, we have $z = z_0 + (z - z_0)$ and

$$\frac{1}{z} = \frac{1}{z_0 + (z - z_0)} = \frac{1}{z_0} \frac{1}{1 + \frac{1}{z_0}(z - z_0)}$$
$$= \frac{1}{z_0} \sum_{n=0}^{\infty} (-1)^n \frac{(z - z_0)^n}{z_0^n}.$$

So can re-write f as a power series in $(z - z_0)$. $\square$

Laurent series

Theorem

If f is holomorphic in the annulus $A = \{z \in \mathbb{C} \colon r < |z - z_0| < R\}$, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

and the series converges in A.

Proof.

The same as in the previous case, but now split in two parts: one for the positive and one for the negative parts. □

Laurent series

Theorem

If f is holomorphic in the annulus $A = \{z \in \mathbb{C} \colon r < |z - z_0| < R\}$, with

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

then

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z - z_0)^{n-1},$$

Laurent series

Example

Consider $f(z) = \frac{1}{z(1-z)}$.
Using

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1,$$

$$f(z) = \sum_{n=-1}^{\infty} z^n.$$

Laurent series

Example

Consider $f(z) = \frac{1}{z(1-z)}$.
Using

$$\frac{1}{1-z} = -\frac{1}{z}\frac{1}{1-1/z} = -\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = -\sum_{n=0}^{\infty} z^{-n-1}$$

We have

$$f(z) = -\sum_{n=0}^{\infty} z^{-n-2}, \qquad \text{for } |z| > 1.$$