Analysis in one complex variable
Lecture 8 – Exercises

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Exercise (Exercise 5.2.10)

Find the Laurent series for $\frac{1}{z^2(1-z)}$ *in the regions: (a)* $0 \leq |z| \leq 1$*; (b)* $|z| \geq 1$.

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For $|z| > 1$ we use the geometric series with a trick to get $1/z$:

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Hence

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2}\sum_{n=1}^{\infty} z^{-n} = \sum_{n=-3}^{\infty} z^{-n}$$

Lemma

If $|b| \leq |a|$,

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Solution.

$$\frac{1}{a-b} = \frac{1}{a}\frac{1}{1-b/a} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n = \sum_{n=0}^{\infty}\frac{b^n}{a^{n+1}}$$

$\square$

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Solution.

Simplify and force the geometric series:

$$\frac{2}{1+z^2} = \frac{2}{(1+iz)(1-iz)} = \frac{1}{(1+iz)} + \frac{1}{(1-iz)}.$$

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Solution.

$$\begin{aligned}
\frac{1}{(1+iz)} &= \frac{1}{(1+i+i(z-1))} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n (z-1)^n}{(1+i)^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n (z-1)^n}{2^{\frac{n+1}{2}} e^{\frac{(n+1)\pi i}{4}}} \qquad = \sum_{n=0}^{\infty} \frac{e^{-\frac{(n+1)\pi i}{4}} (-i)^n (z-1)^n}{2^{\frac{n+1}{2}}}
\end{aligned}$$

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Find the power series expansion of $f(z) = \frac{1}{1+z^2}$ *around the point* $z = 1$*, and find the radius of convergence of this series.*

Solution.

$$\frac{1}{(1-iz)} = \frac{1}{(1-i-i(z-1))}$$
$$= \sum_{n=0}^{\infty} \frac{i^n (z-1)^n}{(1-i)^{n+1}}$$
$$= \sum_{n=0}^{\infty} \frac{(1+i)^{n+1} i^n (z-1)^n}{2^{n+1}}$$