Analysis in one complex variable
Lecture 8 – Singularities

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Singularities

Definition

A holomorphic function $f\colon U \to \mathbb{C}$ has a *singularity* at $z_0 \in \mathbb{C}$ if there is a disc $D \subset \mathbb{C}$ centered at z_0 such that $D\backslash\{z_0\} \subset U$, that is f is defined in all points of the disc except z_0.

Notice that if f has a singularity at a point z_0, then z_0 is *not* in the domain of f.

Singularities

Theorem

Let $f : U \to \mathbb{C}$ be holomorphic and let z_0 be a singularity of f. If f is bounded in a neighbourhood of z_0, then there is a unique continuous extension of f to z_0 and this extension is holomorphic.

Proof.

Say $z_0 = 0$. Since f is analytic in the annulus it has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(\xi)}{\xi^{n+1}} d\xi.$$

Singularities

Proof.

We want to show that $a_n = 0$ for $n < 0$.
Since f is bounded, say $|f| < M$, we have:

$$
\begin{aligned}
|a_n| &= \frac{1}{2\pi}\left|\int_{\partial D_r} \frac{f(\xi)}{\xi^{n+1}} d\xi\right| \\
&\leq \frac{1}{2\pi}\int_{\partial D_r} \frac{|f(\xi)|}{r^{n+1}} d|\xi| \\
&\leq \frac{1}{2\pi}\int_{\partial D_r} \frac{M}{r^{n+1}} d|\xi| \\
&\leq \frac{1}{2\pi} 2\pi r \frac{M}{r^{n+1}} \\
&\leq \frac{M}{r^n} = Mr^{-n}.
\end{aligned}
$$

Singularities

Proof.

We want to show that $a_n = 0$ for $n < 0$.

$$|a_n| \leq M r^{-n}.$$

where r is an arbitrary positive number.
For $n < 0$ we conclude that $a_n = 0$. $\square$

Singularities

Definition

A point z_0 is a *removable singularity* of a meromorphic function $f\colon U \to \mathbb{C}$ if is a singularity of f and one can extend f to z_0 to obtain a function holomorphic at z_0.

That is, a removable singularity is not really a singularity.

Singularities

Example

Let $f\colon U \to \mathbb{C}$ be holomorphic and $f(z_0) = 0$.
Define

$$g\colon U\backslash\{z_0\} \to \mathbb{C}, \qquad g(z) = \frac{f(z)}{z - z_0}.$$

Then g has a removable singularity at z_0.
Indeed, the power series expansion of f is $\sum_{n=1}^{\infty} a_n(z - z_0)^n$, hence

$$g(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^{n-1}$$

Extend $g(z_0) = a_1$.

Singularities

Definition

A point z_0 is a *zero of order m* of a holomorphic function $f : U \to \mathbb{C}$ if f and its first $m-1$ derivatives vanish at z_0 but $\frac{d^m f}{dz^m}|_{z_0} \neq 0$.

Example

Let $f : U \to \mathbb{C}$ be holomorphic and z_0 be a zero of order m of f, then $\frac{f(z)}{(z-z_0)^m}$ has a removable singularity at z_0

Singularities

Exercise

Let $f, g\colon U \to \mathbb{C}$ *be holomorphic and* z_0 *be a zero of order* m *of* f *and a zero of order* $n \leq m$ *of* g *then* $\frac{f}{g}$ *has a removable singularity at* z_0

Poles

Definition

Let $f\colon U \to \mathbb{C}$ have a singularity at z_0. We say that z_0 is a *pole* of f it the Laurent expansion of f at z_0 has finitely many nonzero terms with negative exponent:

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n.$$

Definition

If f has a pole at z_0, the biggest value of n for which $a_{-n} \neq 0$ is the *order* of the pole of f at z_0.

Poles

Definition

A *simple pole* is a pole of order one.

Poles

Definition

A *meromorphic* function on $U \subset \mathbb{C}$ is a holomorphic function defined on U whose singularities are all poles.

Poles

Lemma

If f has a zero of order m at z_0, then $\frac{1}{f}$ has a pole of order m at z_0 and vice versa.

Proof.

Write $f = \sum_{n=m}^{\infty} a_n(z-z_0)^n = (z-z_0)^m \sum_{n=0}^{\infty} a_{n+m}(z-z_0)^n$. Then

$$\frac{1}{f} = \frac{1}{(z-z_0)^m \sum_{n=0}^{\infty} a_{n+m}(z-z_0)^n} = \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} \tilde{a}_n (z-z_0)^n.$$

□

Poles

Lemma

If f has a zero of order m at z_0 and g has a pole of order n at z_0, then

- *gf has a zero of order $m-n$ at z_0 if $m>n$,*
- *gf has a pole of order $n-m$ at z_0 if $n>m$.*