Analysis in one complex variable
Lecture 8 – Singularities

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Recall

- Singularities,
- Removable singularities,
- Poles.

Essential singularities

Definition

Let $f\colon U \to \mathbb{C}$ have a singularity at z_0. We say that z_0 is an *essential singularity* of f it the Laurent expansion of f at z_0 has infinitely many nonzero terms with negative exponent.

Essential singularities

Theorem (Casorati–Weierstrass)

If f has an essential singularity at z_0, then for any disc D around z_0 $f(D\backslash\{z_0\})$ is dense on $\mathbb{C}$.

Proof.

Assume by contradiction that there is D such that $f(D\backslash\{z_0\})$ is not dense on $\mathbb{C}$.
Then there is $\alpha \in \mathbb{C}$ and a disc D' around α such that $D' \not\in f(D\backslash\{z_0\})$.
Consider

$$g\colon D\backslash\{z_0\} \to \mathbb{C}, \qquad g(z) = \frac{1}{f(z) - \alpha}$$

g is bounded, hence z_0 is a removable singularity.

Essential singularities

Proof.

Hence g has at most a zero of some order, say, m at z_0.
Hence $f(z) - \alpha = \frac{1}{g}$ has a pole of order m at z_0.
(contradiction) $\square$

Biholomorphisms of the line

Definition

A *biholomorphism* between U and $V \subset \mathbb{C}$ is a holomorphic map $f\colon U \to V$ with holomorphic inverse.

Lemma

If $f\colon U \to V$ is a holomorphic bijection, then f is a biholomorphism.

Proof.

Indeed, if f' is nowhere zero, then f has a holomorphic inverse, but if f' vanishes at a point then f is not a bijection. □

Biholomorphisms of the line

Lemma (L03P02)

Let $f(z) = a_n z^n + \dots$ with $a_n \neq 0$. Then there is an analytic diffeomorphism φ of a neighbourhood of 0 such that $f \circ \varphi(z) = z^n$.

Biholomorphisms of the line

Theorem

The only biholomorphisms of $\mathbb{C}$ (with itself) are maps of the form $f(z) = az + b$ with $a \neq 0$.

Proof.

Let f be a biholomorphism of $\mathbb{C}$. By composing f with a translation we may assume that $f(0) = 0$.
Hence f maps a neighbourhood U of 0 to a neighbourhood of 0 and, since f is a bijection, f maps the complement of U to the complement of a neighbourhood of 0.
Hence the image of $g\colon D\backslash\{0\} \to \mathbb{C}$, $g(z) := f(1/z)$ not dense.
Therefore g has a pole at 0.

Biholomorphisms of the line

Proof.

If $f(z) = \sum a_n z^n$, then $g(z) = \sum a_n z^{-n}$. Hence f is a polynomial. Since a complex polynomial of degree n has n zero (counted with multiplicity, but f is a bijection, we must have $f(z) = az^n$. Hence $n = 1$. □

The Riemann sphere

Definition

The *Riemann sphere* is the union $S := C \cup \{\infty\}$.
A function $f\colon S \to \mathbb{C}$ is *holomorphic* if $f|_{\mathbb{C}}\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and

$$g\colon \mathbb{C} \to \mathbb{C}, \qquad g(z) = \begin{cases} f(1/z) & \text{for } z \neq 0, \\ f(\infty) & \text{for } z = 0 \end{cases}$$

is holomorphic.

Same idea holds for meromorphic functions.

The Riemann sphere

Lemma

A holomorphic function on the Riemann sphere is constant.

Proof.

Let f be a holomorphic function on S and write its power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Then

$$g(z) = \sum_{n=0}^{\infty} a_n z^{-n}$$

is holomorphic, hence $a_n = 0$ for $n > 0$. $\square$

The Riemann sphere

Lemma

A meromorphic function on the Riemann sphere is a rational function.

Proof.

It is an exercise to show that if $f = p/q$ for p, q polynomials, then f is a meromorphic function on S.
Let f be a meromorphic function on S. Since S is compact and the poles of a meromorphic function must form a discrete set, f has finitely many poles, say $z_1, \ldots, z_n$ and maybe also ∞.
Let m_i be the order of the pole z_i.

The Riemann sphere

Proof.

Consider the function

$$h(z) = f(z)\Pi_i(z - z_i)^{m_i}.$$

Then h has only removable singularities on $\mathbb{C}$ and maybe a pole at ∞. Writing out the series expansion of h we have

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \Rightarrow \qquad h(1/z) = \sum_{n=0}^{\infty} a_n z^{-n}$$

and h has a pole at infinity if and only if $a_n = 0$ for all but finitely many values of n.

The Riemann sphere

Proof.

That is h is a poloynomial.

$$f(z) = \frac{h(z)}{\Pi_i (z - z_i)^{m_i}}$$

is a rational function. □

The Riemann sphere

Lemma

Let p and q be polynomials and consider $f = p/q$. Then

- *if* $\deg(p) > \deg(q)$, f *has a pole at* ∞ *of order* $\deg(p) - \deg(q)$,
- *if* $\deg(p) < \deg(q)$, f *has a zero at* ∞ *of order* $\deg(q) - \deg(p)$.

In particular, the sum of the order of the zeros of a meromorphic function f on S is equal to the sums of the order of the poles of f.

The Riemann sphere

Lemma

The biholomorphisms of S with itself are maps of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad \textit{with } ad - bc \neq 0.$$

Proof.

Case 1) $f(\infty) = \infty$. Then $f : \mathbb{C} \to \mathbb{C}$ is a biholomorphism and $f(z) = az + b = \frac{az+b}{0z+1}$ with $0 \neq a = ad - bc$.

The Riemann sphere

Proof.

Case 2) If $f(\infty) = z_0$ let $g \colon S \to S$ be $g(z) = \frac{1}{z - z_0}$.
Then g is a biholomorphism of S and $g \circ f(\infty) = \infty$, hence

$$cz + d = g \circ f(z) = \frac{1}{f(z) - z_0}.$$

Resolving for $f(z)$:

$$f(z) = \frac{1}{cz + d} + z_0 = \frac{cz_0 z + 1 + dz_0}{cz + d}$$

□