Analysis in one complex variable
Lecture 9 – The Riemann sphere

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Recall

Theorem (Casorati–Weierstrass)

If f has an essential singularity at z_0, then for any disc D around z_0, $f(D\backslash\{z_0\})$ is dense on $\mathbb{C}$.

Remark

If f has a pole at z_0, then compact set $K \subset \mathbb{C}$, there is a disc D around z_0 such that $f(D\backslash\{z_0\}) \cap K = \emptyset$.

Recall

Idea of the proof.

If f has a pole at z_0, then it has a Laurent expansion

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n.$$

The term $|a_{-m}(z - z_0)^{-m}|$ goes to infinity much faster than any other term in the series as $|z - z_0|$ goes to 0, hence determines the behaviour of f in a neighbourhood of z_0. □

Recall – The Riemann sphere

Definition

The *Riemann sphere* is the union $S := \mathbb{C} \cup \{\infty\}$.
A function $f: S \to \mathbb{C}$ is *holomorphic/meromorphic/smooth* if

- $$f|_{\mathbb{C}}: \mathbb{C} \to \mathbb{C} \qquad \text{and}$$
- $$g: \mathbb{C} \to \mathbb{C}, \qquad g(z) = \begin{cases} f(1/z) & \text{for } z \neq 0, \\ f(\infty) & \text{for } z = 0 \end{cases}$$

are holomorphic/meromorphic/smooth.

What happened here?!

Answer

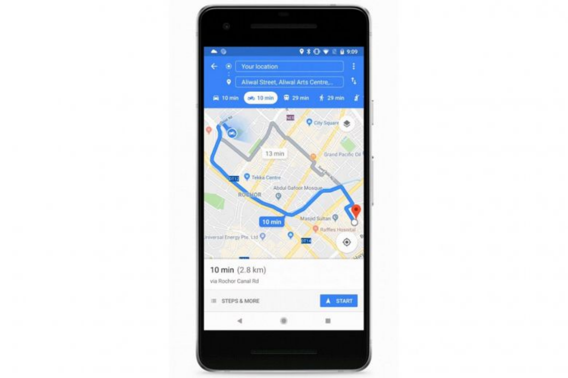
You just came across your second abstract manifold ever.

Question

Second?

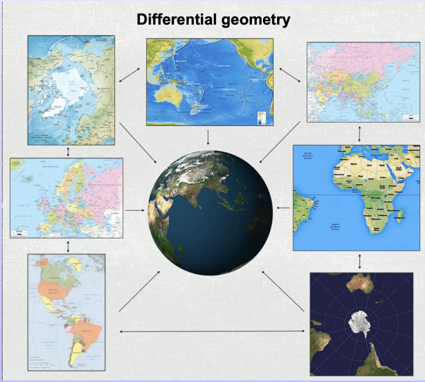
What happened here?!

Some spaces are locally indistinguishable from Euclidean space: you can see any part of it on the screen of your phone.



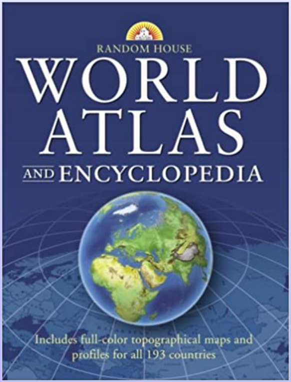
What happened here?!

But when you try to see the space as a whole it turns out it is not $\mathbb{R}^n$.



What happened here?!

For an abstract manifold we have an atlas, but not a “3D” model.



What happened here?!

The Riemann sphere is a manifold with a very thin atlas: it has two pages (charts) and the relation between these charts is

$$w = \frac{1}{z}.$$

What happened here?!

If $f\colon U \to \mathbb{C} \cup \{\infty\}$ is a function we can consider the sets

$$U_\infty = \{z \in U \colon f(z) \neq \infty\} \qquad \text{and} \qquad U_0 = \{z \in U \colon f(z) \neq 0\}$$

To study the behaviour of f we can restrict ourselves to these two sets:

$$f\colon U_\infty \to \mathbb{C}, \qquad \frac{1}{f}\colon U_0 \to \mathbb{C}$$

Meromorphic functions are functions

Given a meromorphic function $f\colon U \to \mathbb{C}$, if $f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$, we rewrite it

$$
\begin{aligned}
f(z) &= a_{-m}(z - z_0)^{-m} \sum_{n=-m}^{\infty} \frac{a_n}{a_{-m}} (z - z_0)^{n+m} \\
&= a_{-m}(z - z_0)^{-m} \left(1 + \frac{a_{-m+1}}{a_{-m}}(z - z_0) + \frac{a_{-m+2}}{a_{-m}}(z - z_0)^2 + \dots\right)
\end{aligned}
$$

Meromorphic functions are functions

Hence

$$\frac{1}{f(z)} = \frac{(z-z_0)^m}{a_{-m}}\left(1+\frac{a_{-m+1}}{a_{-m}}(z-z_0)+\frac{a_{-m+2}}{a_{-m}}(z-z_0)^2+\dots\right)^{-1}$$

So we can define $f(z_0)=\infty$ for all poles z_0.
We have a holomorphic function

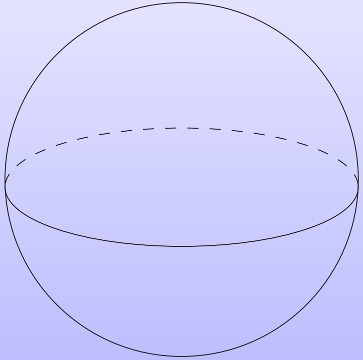
$$f\colon U \to \mathbb{C}\cup\{\infty\}.$$

Meromorphic functions are functions

Notice that by Casorati–Weierstrass theorem, holomorphic functions with essential singularities do not extend continuously to $\mathbb{C} \cup \{\infty\}$.

How round is the Riemann sphere?

We can parametrize a sphere minus one point (north pole) using stereographic projection:



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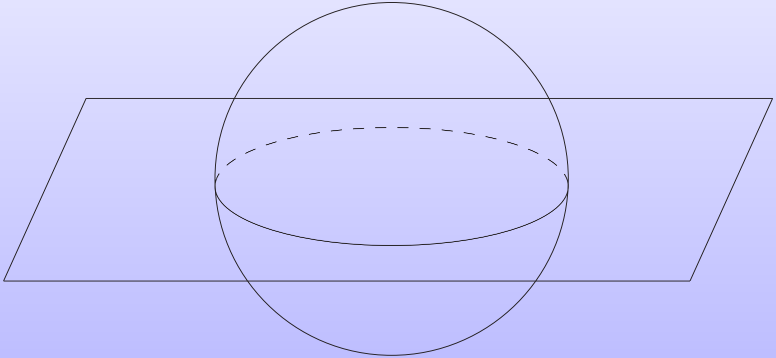

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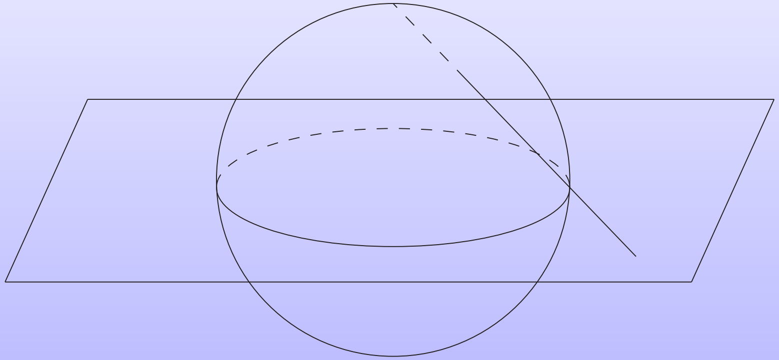

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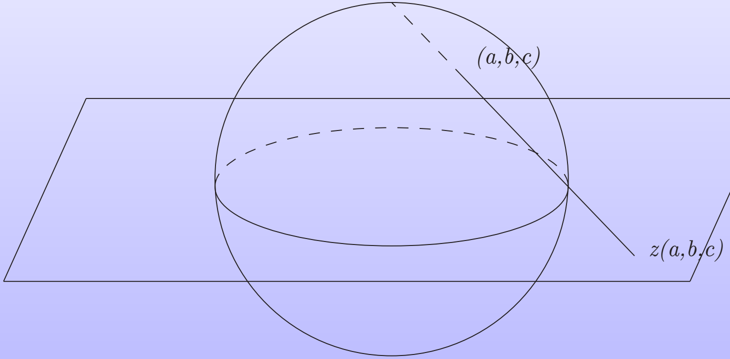

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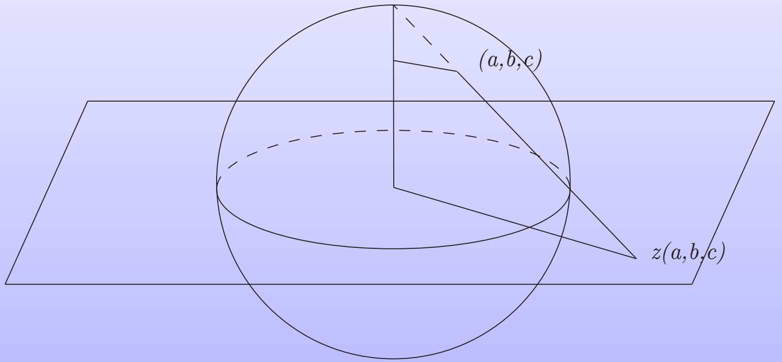

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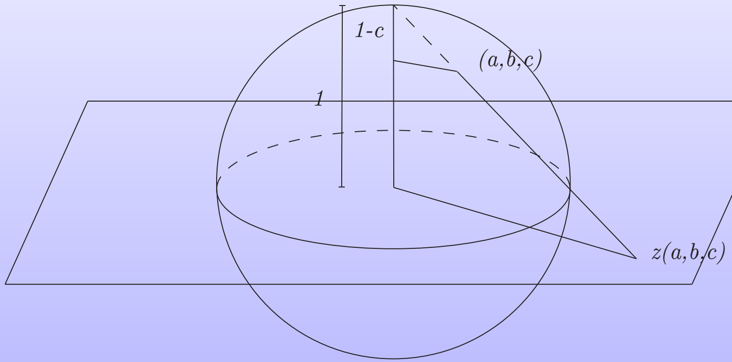

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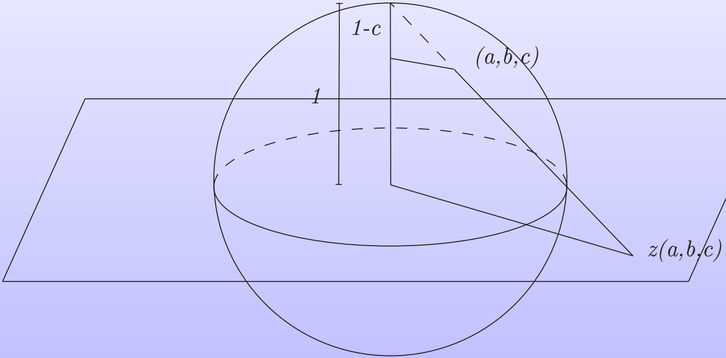
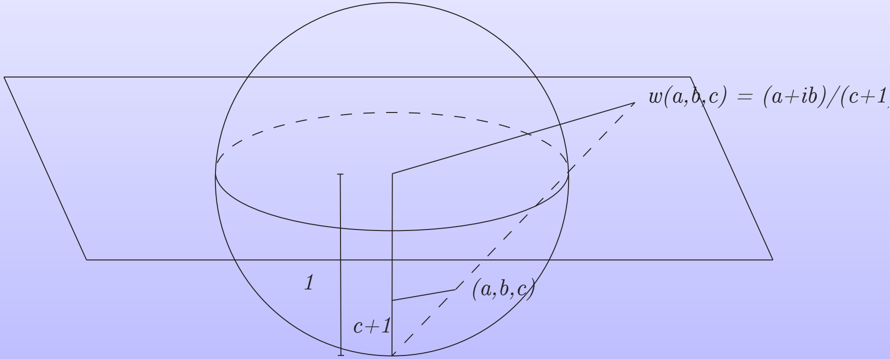


$$z(a,b,c) = \frac{(a,b)}{1-c} = \frac{a+ib}{1-c}$$

How round is the Riemann sphere?

Similarly for the south pole...



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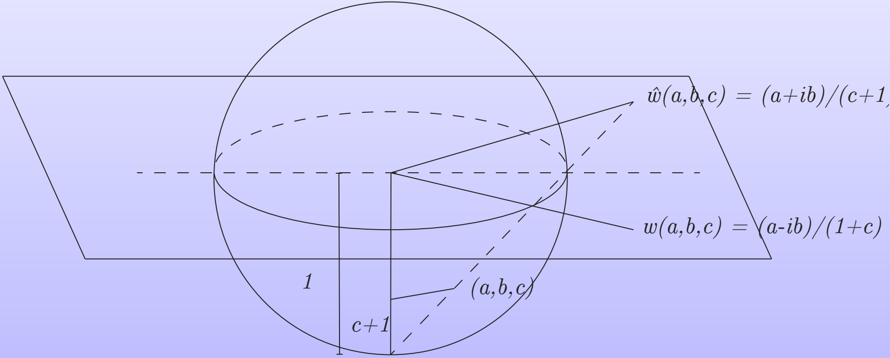

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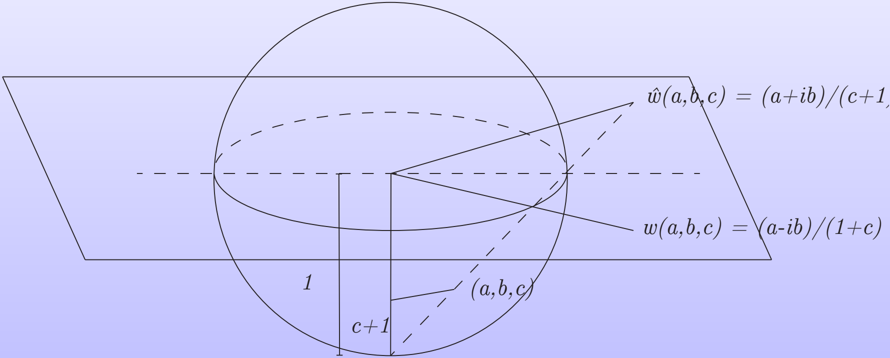

$$w(a,b,c) = \frac{(a,-b)}{1+c} = \frac{a-ib}{1+c}$$

How round is the Riemann sphere?

Notice that

$$zw = \left(\frac{a+ib}{1-c}\right)\left(\frac{a-ib}{1+c}\right) = \frac{a^2+b^2}{1-c^2} = 1.$$

That is $z = \frac{1}{w}$.
The Riemann sphere is just the usual sphere in $\mathbb{R}^3$ parametrized by two stereographic projections.