Analysis in one complex variable
Lecture 9 – Calculus of Residues

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Residues

Definition

The *residue* at z_0 of a holomorphic function with a singularity at z_0 whose Laurent series is given by

$$f(z) = \sum a_n (z - z_0)^n,$$

is number $\operatorname{Res}_{z_0}(f) = a_{-1}$.

Theorem

If z_0 is an isolated singularity of a holomorphic function f and C is a small circle traced counterclockwise around z_0, then

$$\operatorname{Res}_{z_0}(f) = \frac{1}{2\pi i} \int_C f \, dz$$

Residues

Theorem

Let $\gamma\colon [0,1] \to U$ be a loop on U that is null homologous and let $f\colon U \to \mathbb{C}$ be an analytic function on $U \backslash \{z_1, \ldots, z_k\}$. Then

$$\frac{1}{2\pi i} \int_\gamma f\, dz = \sum_{i-1}^{k} m_i \mathrm{Res}\,_{z_i} f,$$

where $m_i = W(\gamma, z_i)$.

- Sometimes we find residues by integrating over simpler curves,
- Sometimes by expanding f at its singularities.

Residue calculus

Lemma

Let f and g be meromorphic with a pole at z_0. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ the coefficients of the Laurent expansion of f and g, respectively. Then

- $\mathrm{Res}_{z_0}(fg) = \sum_k a_{k-1}b_{-k}$
- *If f and g have a simple pole at z_0,* $\mathrm{Res}_{z_0}(fg) = a_{-1}b_0 + a_0b_{-1}$,
- *If f has a simple pole at z_0 and g is holomorphic,* $\mathrm{Res}_{z_0}(fg) = \mathrm{Res}_{z_0}(f)g(z_0)$,
- *If f has a simple pole* $\mathrm{Res}_{z_0}f = \lim_{z\to z_0}(z-z_0)f(z)$,
- *If f is holomorphic, $f(z_0) = 0$ and $f'(z_0) \neq 0$, then* $\mathrm{Res}_{z_0}\frac{1}{f} = \frac{1}{f'(z_0)}$.

Residue calculus

Example

Let $f : \mathbb{C} \to \mathbb{C}$ be $f(z) = \sin z$. Then $f(n\pi) = 0$ for all $n \in \mathbb{Z}$ and $f'(n\pi) = \cos(n\pi) = (-1)^n$. Hence

$$\operatorname{Res}_{n\pi} \frac{1}{f} = \frac{1}{(-1)^n} = (-1)^n.$$

Residue calculus

Example

Compute the residue of $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \frac{z^2}{1-z^2}$ at $z = 1$.
Rewrite f as

$$f(z) = \frac{z^2}{1 - z^2} = \frac{z^2}{(1-z)(1+z)}$$

And observe

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z^2}{(1-z)(1+z)} = \lim_{z\to 1} -\frac{z^2}{(1+z)} = -\frac{1}{2}$$

So f has a simple pole at 1 and its residue is $-1/2$.

Residue calculus

Example

Compute the residue of $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \frac{\sin z}{z^2}$ at $z = 0$.
We know the series expansion for sin, so

$$f(z) = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-1}$$

Hence $\mathrm{Res}_0(f) = 1$

Residue calculus

Example

Let $f\colon U \to \mathbb{C}$ be a holomorphic function defined on an open set U which contains the upper half plane. Assume that there exist numbers $B > 0$ and $c > 0$ such that

$$|f(z)| \leq \frac{B}{|z|^c}, \qquad \forall z \in U.$$

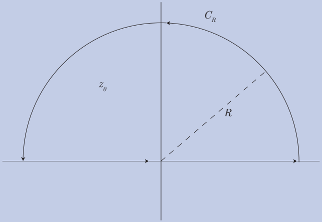
Show that

$$f(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(z)}{z - z_0} dz \qquad \forall z_0 \in \mathbb{R} \times \mathbb{R}_+^\circ.$$

Residue calculus

Example

Consider the C path below where $R > 0$ is an arbitrary large number.



Cauchy integral formula $\Rightarrow \quad f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{-R}^{R} \frac{f(z)}{z - z_0} dz + \int_{C_R} \frac{f(z)}{z - z_0} dz.$$

Residue calculus

Example

For the second integral we have

$$\left|\int_{C_R} \frac{f(z)}{z-z_0}dz\right| \le \int_{C_R} \frac{|f(z)|}{|z-z_0|}|dz|$$
$$\le \int_{C_R} \frac{B}{R^c|z-z_0|}|dz|$$
$$\le \frac{B'}{R^c \cdot R}\pi R$$
$$\le \frac{B'}{R^c}\pi$$

Therefore

$$\lim_{R\to\infty} \int_{C_R} \frac{f(z)}{z-z_0}dz = 0.$$