Analysis in one complex variable
Lecture 10 – More on residues

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Recall

Definition

The *residue* at z_0 of a holomorphic function with a singularity at z_0 whose Laurent series is given by

$$f(z) = \sum a_n(z - z_0)^n,$$

is number $\text{Res}_{z_0}(f) = a_{-1}$.

Theorem

If z_0 is an isolated singularity of a holomorphic function f and C is a small circle traced counterclockwise around z_0, then

$$\text{Res}_{z_0}(f) = \frac{1}{2\pi i} \int_C f \, dz$$

Log

Example

Assume that the Laurent expansion of f has finitely many terms with negative exponent (f has at most a pole at z_0):

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} \quad \Rightarrow \quad f'(z) = \sum_{n=0}^{\infty} a_n (n+m)(z-z_0)^{n+m-1}.$$

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{\sum_{n=0}^{\infty} a_n (n+m)(z-z_0)^{n+m-1}}{\sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}} \\ &= \frac{\sum_{n=0}^{\infty} a_n (n+m)(z-z_0)^{n-1}}{\sum_{n=0}^{\infty} a_n (z-z_0)^{n}} \end{aligned}$$

Log

Example

$$\frac{f'(z)}{f(z)} = (z - z_0)^{-1} \frac{\sum_{n=0}^{\infty} a_n (n + m)(z - z_0)^n}{\sum_{n=0}^{\infty} a_n (z - z_0)^n}$$

$$= (z - z_0)^{-1} g(z),$$

where g is holomorphic and $g(z_0) = m$

Log

Example

Therefore

$$m = \frac{1}{2\pi i} \int_C \frac{f'}{f}\, dz$$

where C is a small circle traced counterclockwise around z_0.
That is,

- if the integral is positive then f has a zero or order m at z_0.
- if the integral is negative then f has a pole or order $-m$ at z_0.

Log

Example

If f is meromorphic on U, C is a simple closed curve inside U with interior region U' and C is null homologous, then

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \sum_{z_i \text{ zero of } f \text{ in } U'} \operatorname{ord}(z_i) - \sum_{w_i \text{ pole of } f \text{ in } U'} \operatorname{ord}(w_i).$$

Log

Theorem (Rouché)

Let f and g be holomorphic functions on U, C is a simple closed curve inside U with interior region U' and C is null homologous. If

$$|f(z) - g(z)| \leq |f(z)| \qquad \text{over } C,$$

then f and g have the same number of zeros in U'.

Log

Proof.

Consider $F = g/f$. Then, by hypothesis, (over C) we have

$$|F \circ C - 1| < 1.$$

Hence

$$0 = \int_{F \circ C} \frac{dz}{z} = \int_C \frac{F'}{F} dz = \int_C \left(\frac{g'}{g} - \frac{f'}{f}\right) dz.$$

□

Log

Example

Applying Rouché's Theorem is a bit of an art form.
For example, how many zeros does $P(z) = z^8 - 5z^3 + z - 2$ have inside in unit disc?
Let $f(z) = -5z^3$, then, over the unit disc we have

$$|f(z) - P(z)| = |z^8 + z - 2| \leq 1 + 1 + 2 \leq 5 = |f(z)|.$$

Hence P has three zeros in the unit disc.