Analysis in one complex variable
Lecture 10 – More on residues

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Recall

Definition

The *residue* at z_0 of a holomorphic function with a singularity at z_0 whose Laurent series is given by

$$f(z) = \sum a_n (z - z_0)^n,$$

is number $\mathrm{Res}_{z_0}(f) = a_{-1}$.

Weird

Example

Let $f = \sum a_n(z - z_0)^n$ be holomorphic with a singularity at z_0 and let $g\colon \mathbb{C} \to \mathbb{C}$ is a local isomorphism (in a neighbourhood of z_0), that is

$$g(w) = \sum_{n=0}^{\infty} b_n(w - w_0)^n,$$

with $b_0 = z_0$ and $b_1 \neq 0$.
Set $z = g(w)$ and compute

$$f \circ g(w) = \sum_n a_n(g(w) - z_0)^n = \sum_n a_n(\sum_{m=1} b_m(w - w_0))^m)^n,$$

Hence $\mathrm{Res}_{w_0}(f \circ g) = a_{-1}b_1 = \mathrm{Res}(f)_{z_0}g'(w_0)$.

Weird

Claim

The residue of a holomorphic function at a singularity is not invariant under (local) isomorphisms.

Weird

The "quantity" dz on the other hand transforms according to the chain rule, so for $z = g(w)$,

$$dz = dg(w) = g'(w)dw$$

Hence $dz|_{z_0} = dz|_{g(w_0)} = g'(w_0)dw|_{w_0}$

$$\text{Res}_{z_0}(f)dz = \frac{\text{Res}_{w_0}(f \circ g)}{g'(w_0)} g'(w_0)dw = \text{Res}_{w_0}(f \circ g)dw.$$

Weird

Claim

The residue of a holomorphic differential at a singularity is invariant under (local) isomorphisms.