Analysis in one complex variable
Lecture 10 – Tricks with integrals

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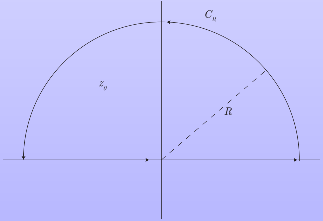
Trick 1

Given a suitable real function $f : \mathbb{R} \to \mathbb{R}$ we want to compute

$$\int_{-\infty}^{\infty} f(x)dx.$$

We will assume that

- f is the restriction to $\mathbb{R}$ of a holomorphic function with singularities $f : \mathbb{C}\backslash\{z_1, \ldots z_n\} \to \mathbb{C}$
- $\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$, where C_R is the semi circle



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In this case, for R large enough,

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i(\sum_{j:\ \mathrm{Im}\,(z_j)>0} \mathrm{Res}_{z_j}(f))$$

Taking the limit:

$$\int_{-R}^{R} f(x)dx = 2\pi i(\sum_{j:\ \mathrm{Im}\,(z_j)>0} \mathrm{Res}_{z_j}(f))$$

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How do we check that $\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$?

Lemma

If there is $B \in \mathbb{R}$ and $c > 0$ such that

$$|f(z)| \leq \frac{B}{|z|^{1+c}}$$

for all z outside a (large) disc centered at the origin, then $\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$.

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Proof.

$$\begin{aligned}\left|\int_{C_R} f(z)dz\right| &\leq \int_{C_R} |f(z)||dz| \\ &\leq \int_{C_R} \frac{B}{R^{1+c}}|dz| \\ &= \frac{B}{R^{1+c}}\pi R \\ &= \frac{B\pi}{R^c}.\end{aligned}$$

□

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Example

Compute

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$$

Here we have the meromorphic function $f(z) = \frac{1}{1+z^4}$.
For $|z| = R > 2$,

$$|1+z^4| > R^4 - 1 = \frac{R^4}{2} + \left(\frac{R^4}{2} - 1\right) > \frac{R^4}{2}$$

Hence $\frac{1}{|1+z^4|} < \frac{2}{|z|^4}$ and by the lemma

$$\lim_{R\to\infty} \int_{C_R} f(z) dz = 0.$$

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Example

Therefore

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = 2\pi i \Big(\sum_{j:\ \mathrm{Im}\,(z_j)>0} \mathrm{Res}_{z_j} f \Big)$$

Notice that $f = 1/g$ for $g = (1 + z^4)$, which is holomorphic.
Hence the poles of f are at the zeros of g and the residue of f is the derivative of g at those points.
Zeros of g are the fourth roots of -1:

$$e^{\frac{2k\pi i}{4}} e^{\frac{\pi i}{4}}, \text{ for } k = 0, 1, 2, 3.$$

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Example

Zeros with positive imaginary part are

$$e^{\frac{2k\pi i}{4}} e^{\frac{\pi i}{4}}, \text{ for } k = 0, 1.$$

$$g'(z) = 4z^3 \Rightarrow \text{Res}_{z_i} f = \frac{1}{4z_i^3}$$

$$\begin{aligned} 2\pi i (\text{Res}_{e^{\frac{\pi i}{4}}} f + \text{Res}_{e^{\frac{3\pi i}{4}}} f) &= 2\pi i \tfrac{1}{4} (e^{-\frac{3\pi i}{4}} + e^{-\frac{\pi i}{4}}) \\ &= \pi i \tfrac{1}{4} (-\sqrt{2} - i\sqrt{2} + \sqrt{2} - i\sqrt{2}) \\ &= \pi i \tfrac{1}{4} (-2i\sqrt{2}) \\ &= \frac{\sqrt{2}}{2} \pi. \end{aligned}$$