Analysis in one complex variable
Lecture 10 – Tricks with integrals

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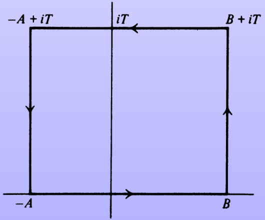
Trick 2

Given a suitable real function $f\colon \mathbb{R} \to \mathbb{R}$ we want to compute

$$\int_{-\infty}^{\infty} f(x)dx.$$

We will assume that

- f is the restriction to $\mathbb{R}$ of a holomorphic function with singularities $f\colon \mathbb{C}\backslash\{z_1, \ldots z_n\} \to \mathbb{C}$
- $\lim_{A,B,T\to\infty} \int_{R^+_{A,B,T}} f(z)\, dz = 0$, where $R^+_{A,B,T}$ are the three sides of the rectangle below outside of the real line



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In this case, for A, B, T large enough,

$$\int_{-A}^{B} f(x)dx + \int_{R^{+}_{A,B,T}} f(z)dz = 2\pi i(\sum_{j:\ \mathrm{Im}\,(z_j)>0} \mathrm{Res}_{z_j}(f))$$

Taking the limit:

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i(\sum_{j:\ \mathrm{Im}\,(z_j)>0} \mathrm{Res}_{z_j}(f))$$

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How do we check that $\lim_{A,B,T\to\infty} \int_{R^+_{A,B,T}} f(z)\, dz = 0$?

Lemma

Let $a > 0$ and Suppose that f is meromorphic function on $\mathbb{C}$ with finitely many poles and without poles in the real line. If there is $K \in \mathbb{R}$ such that

$$|f(z)| \leq \frac{K}{|z|}$$

for all z outside a (large) disc centered at the origin, then $\lim_{A,B,T\to\infty} \int_{R^+_{A,B,T}} e^{iaz} f(z)\, dz = 0$.

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Proof.

Need to compute three integrals but two are similar. For the right hand edge we have $z = B + iy$

$$\begin{aligned}\left|\int_0^T e^{ia(B+iy)} f(B+iy) i dy\right| &\le \int_0^T |e^{iaB-ay} f(B+iy)| dy \\ &\le \int_0^T |e^{-ay}| \frac{K}{B} dy \\ &= \frac{K}{B} \left| \frac{e^{-ay}}{a} \right| \Bigg|_0^T \\ &= \frac{K}{B} \frac{1-e^{-aT}}{a}.\end{aligned}$$

□

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Proof.

For the top edge we have $z = x + iT$ and we take $T = A + B$

$$\begin{aligned} \left| \int_{-A}^{B} e^{ia(x+iT)} f(x+iT) dx \right| &\leq \int_{-A}^{B} |e^{iax-aT} f(x+iT)| dx \\ &\leq \int_{-A}^{B} |e^{-aT}| \frac{K}{T} dx \\ &= e^{-aT} \frac{K}{T} (A+B). \end{aligned}$$

□

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Claim

Let $a > 0$ and Suppose that f is meromorphic function on $\mathbb{C}$ with finitely many poles and without poles in the real line. If there is $K \in \mathbb{R}$ such that

$$|f(z)| \leq \frac{K}{|z|}$$

for all z outside a (large) disc centered at the origin, then

$$\int_{-\infty}^{\infty} e^{iax} f(x)\, dx = 2\pi i \Big(\sum_{j:\ \mathrm{Im}\,(z_j) > 0} \mathrm{Res}_{z_j}\, e^{iaz} f(z) \Big).$$

Notice: $\cos ax = \frac{e^{ia} + e^{-ia}}{2}$ and $\sin ax = \frac{e^{ia} - e^{-ia}}{2i}$