Analysis in one complex variable
Lecture 11 – Tricks with integrals

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Trick 2

Claim

Let $a > 0$ and suppose that f is meromorphic function on $\mathbb{C}$ with finitely many poles and without poles in the real line. If there is $K \in \mathbb{R}$ such that

$$|f(z)| \leq \frac{K}{|z|}$$

for all z outside a (large) disc centered at the origin, then

$$\int_{-\infty}^{\infty} e^{iax} f(x)\, dx = 2\pi i\Big(\sum_{j:\ \mathrm{Im}\,(z_j)>0} \mathrm{Res}_{z_j}\, e^{iaz} f(z)\Big).$$

Notice: $\cos ax = \frac{e^{ia}+e^{-ia}}{2}$ and $\sin ax = \frac{e^{ia}-e^{-ia}}{2i}$

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Example (1)

Compute the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} dx$$

Notice that this is the imaginary part of

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{1+x^2} dx$$

From the previous theorem we have

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{1+x^2} dx = 2\pi i \Big(\sum_{j:\ \mathrm{Im}\,(z_j)>0} \mathrm{Res}_{z_j} \frac{ze^{iz}}{1+z^2} \Big).$$

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Example (1)

$$\frac{ze^{iz}}{1+z^2} = \frac{ze^{iz}}{(z+i)(z-i)}$$

has simple poles at $\pm i$ hence the residue at i is

$$\mathrm{Res}_i \, \frac{ze^{iz}}{1+z^2} = \left. \frac{ze^{iz}}{(z+i)} \right|_{z=i} = \frac{ie^{-1}}{(i+i)} = \frac{1}{2e}.$$

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Example (1)

Returning to our integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} dx = \mathrm{Im}\,(2\pi i \mathrm{Res}_i \frac{z e^{iz}}{1+z^2}) = \mathrm{Im}\,\frac{2\pi i}{2e} = \frac{\pi}{e}.$$

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Example (2)

Compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

Notice that this is the imaginary part of

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$$

But this one has a pole on the real line :-(

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Example (2)

We have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \int_{|x|>\epsilon} \frac{\sin x}{x} dx + \int_{|x|<\epsilon} \frac{\sin x}{x} dx$$

Since $\frac{\sin x}{x} \leq 2$ for $\epsilon > 0$ we have

$$\left| \int_{|x|<\epsilon} \frac{\sin x}{x} dx \right| < 4\epsilon.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\sin x}{x} dx.$$

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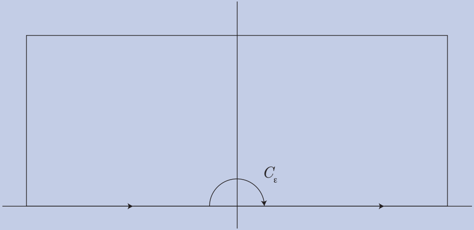
Example (2)

Now we use $\sin x = \operatorname{Im} e^{ix}$ and alter our contour



and get

$$\int_{|x|>\epsilon} \frac{e^{ix}}{x} dx + \int_{C_\epsilon} \frac{e^{iz}}{z} dz = 2\pi i \Big(\sum_{j:\ \operatorname{Im}(z_j)>0} \operatorname{Res}_{z_j} \frac{e^{iz}}{z} \Big) = 0$$

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Example (2)

Over C_ϵ we have $z = \epsilon e^{i(\pi - t)}$ for $t \in [0, \pi]$, so we can compute the second integral directly:

$$\int_{C_\epsilon} e^{iz} \frac{dz}{z} = \int_0^\pi e^{i\epsilon e^{i(\pi-t)}} (-idt) \xrightarrow{\epsilon \to 0} \int_0^\pi -idt = -i\pi$$

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Example (2)

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin x}{x} dx &= \lim_{\epsilon\to 0} \int_{|x|>\epsilon} \frac{\sin x}{x} dx \\ &= \lim_{\epsilon\to 0} \mathrm{Im} \int_{|x|>\epsilon} \frac{e^{ix}}{x} dx \\ &= -\lim_{\epsilon\to 0} \mathrm{Im} \int_{C_\epsilon} \frac{e^{iz}}{z} dz \\ &= -\mathrm{Im}\ -i\pi \\ &= \pi\end{aligned}$$