Analysis in one complex variable
Lecture 11 – Tricks with integrals

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Trick 3

Example

Compute

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta} d\theta.$$

For $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z + z^{-1}}{2i}.$$

Hence we can rewrite the integral

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta} d\theta = \int_{S^1} \frac{1}{1 - \frac{(z-z^{-1})^2}{4}} \frac{dz}{iz}$$

Trick 3

Example

$$\begin{aligned}\int_0^{2\pi} \frac{1}{1+\sin^2\theta} d\theta &= \int_{S^1} \frac{1}{1-\frac{(z-z^{-1})^2}{4}} \frac{dz}{iz} \\ &= \int_{S^1} \frac{4}{4-(z-z^{-1})^2} \frac{dz}{iz} \\ &= \int_{S^1} \frac{4}{(2-z+z^{-1})(2+z-z^{-1})} \frac{dz}{iz} \\ &= \int_{S^1} \frac{4z^2}{(-z^2+2z+1)(z^2+2z-1)} \frac{dz}{iz} \\ &= \int_{S^1} \frac{-4iz}{(-z^2+2z+1)(z^2+2z-1)} dz\end{aligned}$$

Trick 3

Example

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta} d\theta = 2\pi i \sum_{|z_i|<1} (\text{Res}_{z_i} \frac{-4iz}{(-z^2+2z+1)(z^2+2z-1)})$$

Trick 3 – General case

Want to compute

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta) d\theta$$

Use

$$z = e^{i\theta} \Rightarrow \begin{cases} d\theta & = \frac{dz}{iz} \\ \cos\theta & = \frac{z+z^{-1}}{2} \\ \sin\theta & = \frac{z-z^{-1}}{2i} \end{cases}$$

Trick 3 – General case

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta) d\theta = \int_{S^1} Q\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}$$
$$= 2\pi i \sum_{|z_i|<1} \mathrm{Res}_{z_i} \frac{Q\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)}{iz}$$