Analysis in one complex variable
Lecture 11 – Tricks with integrals

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Trick 4 – Mellin Transforms

Definition

Given a suitable function $f\colon \mathbb{R}_+ \to \mathbb{R}$, its Mellin transform is the function

$$\hat{f}\colon \mathbb{R} \supset\!\!\to \mathbb{R}, \qquad \hat{f}(a) = \int_0^\infty f(x) x^{a-1} dx.$$

For this to work we need the integral to converge. We need that there are constants B, and $b > b'$ such that

- $|f(x)x^b| < B$ for large values of x,
- $|f(x)x^{b'}| < B$ for small values of x,

Then $\hat{f}$ is defined on (b', b).

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Assume that

- f is analytic on $\mathbb{C}$ except for a finite number of poles,
- f has no poles on the positive real line,
- a is not an integer.

Then under appropriate conditions on the behavior of f near 0, and when x becomes large, the following holds:

$$\int_0^\infty f(x)x^{a-1}dx = -\frac{\pi e^{-\pi i a}}{\sin \pi a} \sum_{z_i \text{ pole of } f, z\neq 0} \mathrm{Res}_{z_i}(f(z)z^{a-1}).$$

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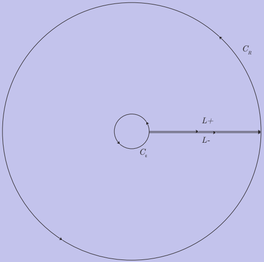
Proof.

Denote $z^a = e^{a \log z}$, where we remove the positive real line to define log (argument is in $(0, 2\pi)$).
Consider the contour:



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Proof.

Compute

$$\int_{C_R} - \int_{C_\epsilon} + \int_{L_+} - \int_{L_-} f(z) z^{a-1} dz$$

we focus on the limits $R \to \infty$ and $\epsilon \to 0$.

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Proof.

Compute (R large) and $a < b$,

$$\begin{aligned}
\left| \int_{C_R} f(z) z^{a-1} dz \right| &\leq \int_{C_R} |f(z)||z|^{a-1}|dz| \\
&\leq \int_{C_R} |f(z)||z|^{b}|z|^{a-b-1}|dz| \\
&\leq BR^{a-b-1} 2\pi R \\
&\leq BR^{a-b} \\
&\to 0 \text{ for } a < b.
\end{aligned}$$

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Proof.

Compute (ϵ small)

$$\begin{aligned}\left|\int_{C_\epsilon} f(z) z^{a-1} dz\right| &\leq \int_{C_\epsilon} |f(z)||z|^{a-1}|dz| \\ &\leq \int_{C_\epsilon} |f(z)||z|^{b'}|z|^{a-b'-1}|dz| \\ &\leq B\epsilon^{a-b'-1} 2\pi\epsilon \\ &\leq C\epsilon^{a-b'} \\ &\to 0 \text{ for } a > b'.\end{aligned}$$

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Proof.

For $a \in (b', b)$,

$$\int_{C_R} - \int_{C_\epsilon} f(z) z^{a-1} dz \to 0.$$

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Proof.

L_+ and $a \in (0,2)$ we get $\hat{f}(a)$ (as $R \to \infty$ and $\epsilon \to 0$).
For L_- we get $z = re^{2\pi i}$.

$$\begin{aligned}\int_0^\infty f(x)x^{a-1}dx &= \int_0^\infty f(z)e^{(a-1)(\log r+2\pi i)}e^{2\pi i}dr \\ &= e^{2\pi i a}\int_0^\infty f(z)r^{a-1}dr \\ &= e^{2\pi i a}\hat{f}(a).\end{aligned}$$

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Proof.

Hence

$$\int_{L+} - \int_{L-} f(z) z^{a-1} dz = \hat{f}(a)(1 - e^{2\pi i a})$$
$$= \hat{f}(a) e^{\pi i a}(e^{-\pi i a} - e^{\pi i a})$$
$$= -\hat{f}(a) e^{\pi i a} 2i \sin(\pi i a)$$

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Proof.

Putting all of this together we obtain

$$2\pi i\Big(\sum_{z_i \text{ pole of } f, z\neq 0} \mathrm{Res}_{z_i}(f(z)z^{a-1})\Big) = \lim_{R\to\infty,\epsilon\to 0} \int_{L+} - \int_{L-} f(z)z^{a-1}dz$$
$$= -\hat{f}(a)e^{\pi i a}2i\sin(\pi i a)$$

Hence

$$\hat{f}(a) = -\frac{\pi e^{-\pi i a}}{\sin(\pi i a)}\Big(\sum_{z_i \text{ pole of } f, z\neq 0} \mathrm{Res}_{z_i}(f(z)z^{a-1})\Big).$$

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