Analysis in one complex variable
Lecture 12 – The disc

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Jun 2020
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Recall

Definition

- Let $U, V \subset \mathbb{C}$ be open subsets. An *isomorphism* between U and V is holomorphic bijection $\varphi\colon U \to V$ with holomorphic inverse.
- An *automorphism* of $U \subset \mathbb{C}$ is an isomorphism $\varphi\colon U \to U$.

Remark

If $\varphi\colon U \to V$ is holomorphic and bijective, then is is an isomorphism.

The Riemann mapping Theorem

Theorem (Riemann mapping Theorem)

Let M be a connected and simply-connected one (complex) dimensional space. Then M is isomorphic to exactly one of the following

- *The Riemann sphere, $\mathbb{C} \cup \{\infty\}$,*
- *The complex plane, $\mathbb{C}$,*
- *The unit disc, $D \subset \mathbb{C}$.*

Remark

No open proper subset $U \subsetneq \mathbb{C}$ is isomorphic to $\mathbb{C}$.

Recall

Theorem (L08P02)

- *The automorphisms of* $\mathbb{C} \cup \{\infty\}$ *are maps of the form* $z \mapsto \frac{az+b}{cz+d}$ *with* $ad - bc \neq 0$,
- *The automorphisms of* $\mathbb{C}$ *are maps of the form* $z \mapsto az + b$ *with* $a \neq 0$.

Aim

Automorphisms of D?

Schwartz Lemma

Theorem (Schwartz Lemma)

Let $f : D \to D$ *be a holomorphic function of D to D with* $f(0) = 0$. *Then*

- $|f(z)| \leq |z|$ *for all* $z \in D$,
- *If* $|f(z)| = |z|$ *for some* $z \in D$, $z \neq 0$, *then there is a with* $|a| = 1$ *such that*
$$f(z) = az.$$

Schwartz Lemma

Proof.

- Denote $r = |z|$. Since f is holomorphic and $f(0) = 0$ it follows that $\frac{f(z)}{z}$ is analytic.
- From $|f(z)| \leq 1$ we obtain

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$$

- Taking the limit as $r \to 1$ we have

$$\left|\frac{f(z)}{z}\right| \leq 1 \qquad \forall z \in \partial D.$$

Schwartz Lemma

Proof.

- By the maximum modulus theorem,

$$\left|\frac{f(z)}{z}\right| \leq 1 \qquad \forall z \in D.$$

Hence

$$|f(z)| \leq |z|.$$

$\square$

Schwartz Lemma

Proof.

- By the maximum modulus theorem, if

$$\left|\frac{f(z)}{z}\right| = 1 \qquad \text{for some } z \in D.$$

then

$$\frac{f(z)}{z} = a.$$

□

Schwartz Lemma v2

Theorem

Let $f : D \to D$ *be holomorphic with* $f(0) = 0$*. Then*

- $|f'(0)| \leq 1$
- *If* $|f'(0)| = 1$ *then* $f(z) = az$ *for some* a *with* $|a| = 1$*.*

Schwartz Lemma v2

Proof.

Notice that $f'(0) = \frac{f(z)}{z}\Big|_{z=0}$.
So, from the proof of the Schwartz Lemma $|f'(0)| \leq 1$ and if $|f'(0)| = 1$ then $f(z) = az$ for some a with $|a| = 1$. □

Automorphisms of D

Theorem

The automophisms of the unit disc, D are maps of the form

$$g_{w,\theta}(z) = e^{i\theta}\frac{w - z}{1 - \overline{w}z}.$$

for $w \in D$ *and* $\theta \in [0, 2\pi)$.

Automorphisms of D

Proof.

First we check that $g_{w,\theta}$ is an automorphism of the disc.
For $|z| = 1$, we have $z^{-1} = \bar{z}$ and

$$\begin{aligned} |g_{w,\theta}(z)| &= \left| e^{i\theta} \frac{w - z}{1 - \overline{w}z} \right| \\ &= |z| \frac{|wz^{-1} - 1|}{|1 - \overline{w}z|} \\ &= \frac{|w\bar{z} - 1|}{|1 - \overline{w}z|} \\ &= 1. \end{aligned}$$

By the maximum modulus principle $|g_{w,\theta}(z)| < 1$ for $|z| < 1$.

$$g_{w,\theta} \colon D \to D.$$

Automorphisms of D

Proof.

Next notice that $g_{w,\theta}$ is a composition of a rotation with $g_{w,0}$ and a direct computation yields

$$g_{w,0} \circ g_{w,0} = \mathrm{Id}$$

So the $g_{w,0}(D) = D$ and $g_{w,0}$ is an automorphism of the disc.

Automorphisms of D

Proof.

- Given $f\colon D \to D$ an automorphism, let $w = f^{-1}(0)$ and consider $h = f \circ g_{w,0}\colon D \to D$.
- $h(0) = 0$ and $h\colon D \to D \Rightarrow |h(z)| \leq |z| \Rightarrow |u| \leq |h^{-1}(u)|$.
- But h^{-1} satisfies the same properties, hence $|h^{-1}(z)| \leq |z| \Rightarrow |z| \leq |h(z)| \Rightarrow h(z) = e^{i\theta}z$ for some θ.
- $f(z) = h \circ g_{w,0}(z) = g_{w,\theta}(z)$.

□