Analysis in one complex variable
Lecture 12 – Möbius transformations

Gil Cavalcanti

Utrecht University

Jun 2020
Utrecht

Möbius Transformations

Definition

A *Möbius transformation* or a *fractional linear transformation* is a map $f \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of the form

$$f(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$.

Example

$$z \mapsto z, \qquad z \mapsto z + 1, \qquad z \mapsto 1/z \qquad z \mapsto az.$$

Möbius Transformations

Lemma

The set Möb, of Möbius transformations, forms a group under composition.

Proof.

Indeed, they are the automorphisms of the Riemann sphere. □

Möbius Transformations

Lemma

The map $\varphi\colon GL(2;\mathbb{C}) \to$ *Möb given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\varphi} \frac{az+b}{cz+d}$$

is a group homomorphism.
The kernel of φ *are the matrices of the form* $\lambda \mathrm{Id}$.

Proof.

Direct computation :-(□

Möbius Transformations

Corollary

If

$$\frac{az+b}{cz+d} = \frac{a'z+b'}{c'z+d'} \qquad \forall z,$$

then there is $\lambda \in \mathbb{C}\setminus\{0\}$ *such that* $a = \lambda a'$, $b = \lambda b'$, *etc.*

Möbius Transformations

Definition

- *Translation by b* is the map $z \mapsto z + b$.
- *Multiplication by a* is the map $z \mapsto az$.
- *Inversion* is the map $z \mapsto z^{-1}$

Lemma

Every Möbius transformation is a composition of scalings, translations and inversion on the circle.

Möbius Transformations

Proof.

$$
\begin{aligned}
f(z) &= \frac{az+b}{cz+d} \\
&= \frac{a}{c}\frac{z+b/a}{z+d/c} \\
&= \frac{a}{c}\left(\frac{z+d/c}{z+d/c} + \frac{b/a-d/c}{z+d/c}\right) \\
&= \frac{a}{c}\left(1 + (b/a-d/c)\frac{1}{z+d/c}\right)
\end{aligned}
$$

with $ad - bc \neq 0$. □

Möbius Transformations

Theorem

Möbius transformations map circles and lines on $\mathbb{C}$ to circles and lines one $\mathbb{C}$.

Proof.

- If two maps send circlines to circlines, so does their composition.
- The statement is clearly true for scalings and translations.
- We only need to prove that inversion satisfies this property.

Möbius Transformations

Proof.

If $(w - w_0)(\overline{w} - \overline{w_0}) = r^2$ and

$$w = \frac{az+b}{cz+d}$$

we get

$$(\frac{az+b}{cz+d} - w_0)(\frac{\overline{az}+\overline{b}}{\overline{cz}+\overline{d}} - \overline{w_0}) = r^2$$

$$((az+b)-(cz+d)w_0)((\overline{az}+\overline{b})-(\overline{cz}+\overline{d})\overline{w_0}) = (cz+d)(\overline{cz}+\overline{d})r^2$$

$$(a\bar{a} - a\overline{c}\overline{w_0} - \bar{a}cw_0 + c\bar{c}w_0\overline{w_0} - c\bar{c}r^2)z\bar{z} + \text{ linear polynomial } = 0.$$

$$\lambda(x^2+y^2) + \alpha x + \beta y + \gamma = 0$$

□

Fixed points

We want to prove that a Möbius transformation is determined by what it does to any three points in $\mathbb{C} \cup \{\infty\}$.

Definition

A *fixed point* of a map $f\colon U \to U$ is a point $x \in U$ for which $f(x) = x$.

Fixed points

Lemma

If a Möbius transformation

$$z \overset{f}{\mapsto} \frac{az+b}{cz+d}$$

fixes $0, 1, \infty$ *it is the identity.*

Proof.

- From $f(0) = 0$ we get $b = 0$.
- From $f(\infty) = \infty$ we get $c = 0$.
- From $f(1) = 1$ we get $a = d$.

□

Fixed points

Theorem

Given two sets of three points, (z_1, z_2, z_3) and (w_1, w_2, w_3), there is a unique Möbius transformation w such that $w(z_i) = w_i$.

Proof.

Define w implicitly by

$$\frac{z - z_1}{z - z_2}\frac{z_3 - z_2}{z_3 - z_1} = \frac{w - w_1}{w - w_2}\frac{w_3 - w_2}{w_3 - w_1}.$$

This proves existence.

Fixed points

Proof.

In particular,

$$f(z) = \frac{z - z_1}{z - z_2}\frac{z_3 - z_2}{z_3 - z_1}$$

maps z_1 to 0, z_2 to ∞ and z_3 to 1.
If w and $\tilde{w}$ map z_i to w_i, then

$$f \circ \tilde{w}^{-1} \circ w \circ f^{-1}$$

has 0, 1 and ∞ as fixed points, hence is the identity.

$$f \circ \tilde{w}^{-1} \circ w \circ f^{-1} = \mathrm{Id} \Rightarrow \tilde{w}^{-1} \circ w = f^{-1} \circ f = \mathrm{Id} \Rightarrow w = \tilde{w}.$$

□

Fixed points

Corollary

If a Möbius transformation fixes three points, it is the identity.

Proof.

The identity is a Möbius transformation that also fixes the same three points. The result follows by uniqueness. □