# Analysis in one complex variable Lecture 12 – Möbius transformations

## Gil Cavalcanti

Utrecht University

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L12P03 - Möbius transformations

Cavalcanti

### Definition

A *Möbius transformation* or a *fractional linear transformation* is a map  $f : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$  of the form

$$f(z) = \frac{az+b}{cz+d}$$

with  $ad - bc \neq 0$ .

Example

$$z \mapsto z, \qquad z \mapsto z+1, \qquad z \mapsto 1/z \qquad z \mapsto az.$$

#### Lemma

*The set Möb, of Möbius transformations, forms a group under composition.* 

#### Proof.

Indeed, they are the automorphisms of the Riemann sphere.

#### Lemma

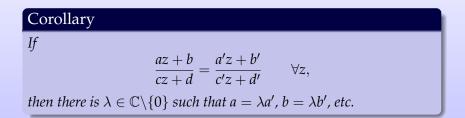
*The map*  $\varphi$ : *GL*(2;  $\mathbb{C}$ )  $\rightarrow$  *Möb given by* 

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\varphi} \frac{az+b}{cz+d}$$

is a group homomorphism. The kernel of  $\varphi$  are the matrices of the form  $\lambda$ Id.

### Proof.

Direct computation :-(



### Definition

- *Translation by b* is the map  $z \mapsto z + b$ .
- *Multiplication by a* is the map  $z \mapsto az$ .
- *Inversion* is the map  $z \mapsto z^{-1}$

#### Lemma

*Every Möbius transformation is a composition of scalings*, *translations and inversion on the circle*.

### Proof.

$$f(z) = \frac{az+b}{cz+d}$$
  
=  $\frac{a}{c}\frac{z+b/a}{z+d/c}$   
=  $\frac{a}{c}\left(\frac{z+d/c}{z+d/c} + \frac{b/a-d/c}{z+d/c}\right)$   
=  $\frac{a}{c}\left(1+(b/a-d/c)\frac{1}{z+d/c}\right)$ 

with  $ad - bc \neq 0$ .

L12P03 - Möbius transformations

### Theorem

*Möbius transformations map circles and lines on*  $\mathbb{C}$  *to circles and lines one*  $\mathbb{C}$ *.* 

### Proof.

- If two maps send circlines to circlines, so does their composition.
- The statement is clearly true for scalings and translations.
- We only need to prove that inversion satisfies this property.

### Proof.

If 
$$(w - w_0)(\overline{w} - \overline{w_0}) = r^2$$
 and

$$w = \frac{az+b}{cz+d}$$

#### we get

$$(\frac{az+b}{cz+d}-w_0)(\frac{\overline{az}+\overline{b}}{\overline{cz}+\overline{d}}-\overline{w_0}) = r^2$$

$$((az+b)-(cz+d)w_0)((\overline{az}+\overline{b})-(\overline{cz}+\overline{d})\overline{w_0}) = (cz+d)(\overline{cz}+\overline{d})r^2$$

$$(a\overline{a}-a\overline{cw_0}-\overline{a}cw_0+c\overline{c}w_0\overline{w_0}-c\overline{c}r^2)z\overline{z} + \text{ linear polynomial } = 0.$$

$$\lambda(x^2+y^2)+\alpha x+\beta y+\gamma = 0$$

#### Cavalcanti

We want to prove that a Möbius transformation is determined by what it does to any three points in  $\mathbb{C} \cup \{\infty\}$ .

### Definition

A *fixed point* of a map  $f: U \to U$  is a point  $x \in U$  for which f(x) = x.

#### Lemma

#### If a Möbius transformation

$$z \stackrel{f}{\mapsto} \frac{az+b}{cz+d}$$

fixes  $0, 1, \infty$  it is the identity.

#### Proof.

- From f(0) = 0 we get b = 0.
- From  $f(\infty) = \infty$  we get c = 0.
- From f(1) = 1 we get a = d.

#### Theorem

Given two sets of three points,  $(z_1, z_2, z_3)$  and  $(w_1, w_2, w_3)$ , there is a unique Möbius transformation w such that  $w(z_i) = w_i$ .

### Proof.

Define *w* implicitly by

$$\frac{z-z_1}{z-z_2}\frac{z_3-z_2}{z_3-z_1} = \frac{w-w_1}{w-w_2}\frac{w_3-w_2}{w_3-w_1}$$

This proves existence.

### Proof.

In particular,

$$f(z) = \frac{z - z_1}{z - z_2} \frac{z_3 - z_2}{z_3 - z_1}$$

maps  $z_1$  to 0,  $z_2$  to  $\infty$  and  $z_3$  to 1. If w and  $\tilde{w}$  map  $z_i$  to  $w_i$ , then

$$f \circ \tilde{w}^{-1} \circ w \circ f^{-1}$$

has 0, 1 and  $\infty$  as fixed points, hence is the identity.

$$f \circ \tilde{w}^{-1} \circ w \circ f^{-1} = \mathrm{Id} \Rightarrow \tilde{w}^{-1} \circ w = f^{-1} \circ f = \mathrm{Id} \Rightarrow w = \tilde{w}$$

### Corollary

If a Möbius transformation fixes three points, it is the identity.

### Proof.

The identity is a Möbius transformation that also fixes the same three points. The result follows by uniqueness.  $\hfill \Box$