Analysis in one complex variable
Lecture 13 – Geodesics

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Measuring distances – Recall

In L04p04, we argued that given $\gamma\colon [0,1] \to \mathbb{R}^2$

$$\text{length}(\gamma) = \int_0^1 \|\gamma'(t)\||dt|.$$

Lemma

If $\tilde{\gamma} = \gamma \circ \sigma$, where $\sigma\colon [0,1] \to [0,1]$ is a diffeomorphism then

$$\text{length}(\gamma) = \text{length}(\tilde{\gamma}),$$

that is, the length of γ is independent of the parametrization.

Measuring distances

Definition

For $A, B \in V^*$, $v, w \in V$ we have

$$A \odot B(v,w) = \frac{1}{2}(A(v)B(w) + B(v)A(w)) = B \odot A(v,w).$$

It is worth introducing the symmetric inner product:

$$g = dx \odot dx + dy \odot dy.$$

Measuring distances – Recall

Given $X = x_1e_1 + x_2e_2$ and $Y = y_1e_1 + y_2e_2$, we have

$$\begin{aligned} g(X, Y) &= dx(x_1e_1 + x_2e_2)dx(y_1e_1 + y_2e_2) + dy(x_1e_1 + x_2e_2)dy(y_1e_1 + y_2e \\ &= x_1y_1 + x_2y_2 \\ &= \langle X, Y \rangle \end{aligned}$$

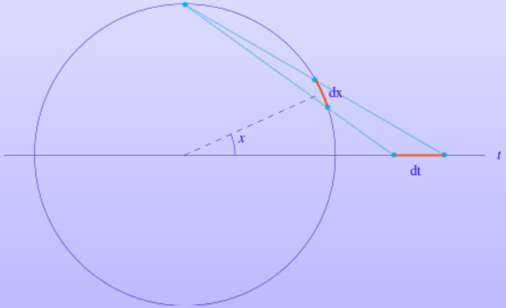
Putting these together

$$\text{length}(\gamma) = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt.$$

Measuring distances – The sphere

For the sphere with stereographic projection, we have

$$g = \frac{4}{(1 + x^2 + y^2)^2}(dx \odot dx + dy \odot dy).$$



Curvature

If we endow $\mathbb{R}^2$ with a metric of the form

$$g = f(x,y)(dx \odot dx + dy \odot dy),$$

the curvature is given by

$$K(x,y) = -\frac{1}{2f(x,y)}(\triangle \log f)$$

where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Measuring distances – Curvature

$$K(x,y) = -\frac{1}{2f(x,y)}(\triangle \log f)$$

For the plane: $f(x,y) = 1$, hence $\triangle \log f = 0 \Rightarrow K = 0$.

Curvature

$$K(x,y) = -\frac{1}{2f(x,y)}(\triangle \log f)$$

For the sphere: $f(x,y) = \frac{4}{(1+x^2+y^2)^2}$, hence

$$\begin{aligned}
\triangle \log f &= \triangle \log \frac{4}{(1+x^2+y^2)^2} \\
&= \triangle \log 4 - 2\triangle \log(1+x^2+y^2) \\
&= -2\left(\frac{\partial}{\partial x}\frac{2x}{1+x^2+y^2} + \frac{\partial}{\partial y}\frac{2y}{1+x^2+y^2}\right) \\
&= -2\left(\frac{4(1+x^2+y^2) - 4x^2 - 4y^2}{(1+x^2+y^2)^2}\right) \\
&= -8\frac{1}{(1+x^2+y^2)^2}
\end{aligned}$$

Curvature

$$
\begin{aligned}
K(x,y) &= -\frac{1}{2f(x,y)}(\triangle \log f) \\
&= -\frac{(1+x^2+y^2)^2}{8}\left(-8\frac{1}{(1+x^2+y^2)^2}\right) \\
&= 1
\end{aligned}
$$

as if it was meant to be...

Curvature

Recap

- *Sphere:* $K = 1$
- *Plane:* $K = 0$
- *???:* $K = -1$
- *???* $\neq \mathbb{RP}^2$, $\mathbb{M}$, $\mathbb{CP}^1$, *saddle* ($z = x^2 - y^2$).

Curvature

Exercise

Consider the unit disc, $\mathbb{D}$, with the following inner product

$$g = \frac{4}{(1 - x^2 - y^2)^2}(dx \odot dx + dy \odot dy).$$

Compute the curvature of the disc.

Curvature

Exercise

Consider the upper half plane, $\mathbb{H}$, *with the following inner product*

$$g = \frac{1}{y^2}(dx \odot dx + dy \odot dy).$$

Compute its curvature.

Complex numbers

$$x = \frac{z + \bar{z}}{2} \qquad y = \frac{z - \bar{z}}{2i}$$

$$dx = \frac{dz + d\bar{z}}{2} \qquad dy = \frac{dz - d\bar{z}}{2i}$$

$$dx \odot dx = \frac{(dz + d\bar{z})^2}{4} = \frac{dz^2 + 2dz \odot d\bar{z} + d\bar{z} \odot d\bar{z}}{4}$$

$$dy \odot dy = -\frac{(dz - d\bar{z})^2}{4} = -\frac{dz^2 - 2dz \odot d\bar{z} + d\bar{z} \odot d\bar{z}}{4}$$

$$dx \odot dx + dy \odot dy = dz \odot d\bar{z}$$

Complex numbers

Metric on $\mathbb{C}$:

$$g = dz \odot d\bar{z}$$

Metric on the sphere, $\mathbb{C} \cup \{\infty\}$:

$$g = \frac{4}{(1+|z|^2)^2} dz \odot d\bar{z}$$

Metric on $\mathbb{D}$:

$$g = \frac{4}{(1-|z|^2)^2} dz \odot d\bar{z}$$

Metric on $\mathbb{H}$:

$$g = \frac{-4}{(z-\bar{z})^2} dz \odot d\bar{z}$$

Upper half plane

Consider the metric on $\mathbb{H}$,

$$g = \frac{1}{y^2} dz \odot d\bar{z},$$

and the transformation $f : \mathbb{H} \to \mathbb{H}$, $z = f(w)$,

$$z = \frac{aw + b}{cw + d}$$

with $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$.

Upper half plane

$$dz = \frac{a(cw+d) - c(aw+b)}{(cw+d)^2} dw$$

$$d\bar{z} = \frac{a(c\bar{w}+d) - c(a\bar{w}+b)}{(c\bar{w}+d)^2} d\bar{w}$$

with $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$.

Upper half plane

Plug it all in and compute :-(

$$\frac{-4}{z-\bar{z}}(dz \odot d\bar{z}) = \frac{-4}{w-\bar{w}}(dw \odot d\bar{w}).$$

(Shortcut: df preserves angles: only need to check that determinant is 1)
Cute, but what does it mean?

$$f^*g = g.$$

Upper half plane

Given a path $\gamma\colon [0,1] \to \mathbb{H}$ and a Mobius transformation, f, as above consider the path $\tilde{\gamma} = f \circ \gamma$. Then

$$\begin{aligned} \text{length}(\tilde{\gamma}) &= \int_0^1 \sqrt{g\left(\frac{df \circ \gamma}{dt}, \frac{df \circ \gamma}{dt}\right)} dt \\ &= \int_0^1 \sqrt{g(df(\gamma'), df(\gamma'))} dt \\ &= \int_0^1 \sqrt{f^* g(\gamma', \gamma')} dt \\ &= \int_0^1 \sqrt{g(\gamma', \gamma')} dt \\ &= \text{length}(\gamma) \end{aligned}$$

That is, the map $f(w) = \frac{aw+b}{cw+d}$ preserves lengths of paths (isometry).

Upper half plane – Geodesics

Question

What is the shortest path between $(0, a)$ and $(0, b)$?

- Say $\gamma\colon [0,1] \to \mathbb{H}$, $\gamma(t) = (x(t), y(t))$, connects $(0, a)$ to $(0, b)$.
- Consider the path $\tilde{\gamma}(t) = (0, y(t))$.

$$\text{length}(\gamma) = \int_0^1 \sqrt{\frac{(x')^2 + (y')^2}{y^2}} dt \geq \int_0^1 \sqrt{\frac{(y')^2}{y^2}} dt = \text{length}(\tilde{\gamma})$$

- The path of shortest length is the vertical line segment between $(0, a)$ to $(0, b)$.

Upper half plane – Geodesics

Question

What is the shortest path between z_0 *and* z_1*?*

Remark

If γ *is the shortest path between* z_0 *and* z_1 *and* f *is an isometry, then* $f \circ \gamma$ *is the path of shortest length between* $f(z_0)$ *and* $f(z_1)$.

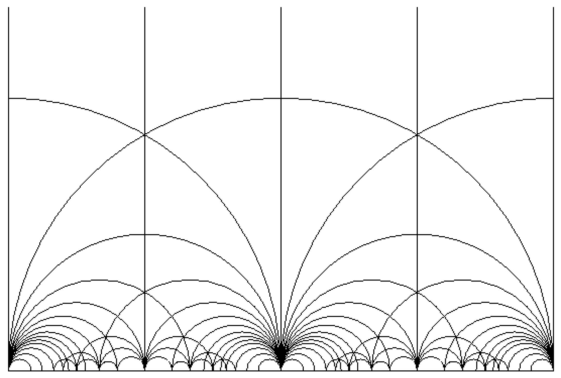
Upper half plane – Geodesics

- Möbius transformations send circlines to circlines,
- Möbius transformations preserve angles of intersections between paths,
- Möbius transformations are determined by what they do to any three points.

Upper half plane – Geodesics

Example

The geodesics on the upper half plane are (part of) vertical lines or semicircles centered at the real line.



Upper half plane – Geodesics

Example

The geodesics on the upper half plane are (part of) vertical lines or semicircles centered at the real line.

What is your name, again?

- $\mathbb{C}$ is the *complex plane*,
- $\mathbb{C} \cup \{\infty\}$ is the *Riemann sphere*,
- $\mathbb{D}$ is the *Poincaré disc*,
- $\mathbb{H}$ is the *hyperbolic or Lobachevsky plane*.

The disc– Geodesics

Exercise

Describe the isometries and the geodesics of the Poincaré disc.