Analysis in one complex variable
Lecture 14 – Harmonic functions

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Jun 2020
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Harmonic functions

Definition

In $\mathbb{R}^n$ the *Laplacian* is the second order linear operator

$$\triangle : C^2(\mathbb{R}^n) \to C^0(\mathbb{R}^n), \qquad \triangle u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u.$$

A function u is *harmonic* if $\triangle u = 0$.

Definition

$\triangle u = 0$ is the Laplace equation.

$-\triangle u = f$ is the Poisson equation.

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Definition

The *gradient* of a function $u\colon \mathbb{R}^n \to \mathbb{R}$ is the vector field

$$\nabla u = \sum \frac{\partial u}{\partial x_i} e_i.$$

The divergent of a vector field $X = \sum X_i e_i$ is

$$\nabla \cdot X = \sum \frac{\partial X_i}{\partial x_i}.$$

We have $\triangle = \nabla \cdot \nabla$.

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Theorem (Gauss/Green/Stokes)

Let $U \subset \mathbb{R}^n$ be an open set with smooth boundary, and let X be a vector field on U, then

$$\int_U \nabla \cdot X = \int_{\partial U} X \cdot \nu dS,$$

where ν is the outward pointing unit vector for ∂U and dS is the area element for ∂U.

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Exercise (Green's formulas)

Let $U \subset \mathbb{R}^n$ be an open set with smooth boundary, and let $u, v \colon U \to \mathbb{R}$ be real functions on U. Then

- $\int_U \triangle u \, dx = \int_{\partial U} \nabla u \cdot \nu dS = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS$,
- $\int_U (\nabla u \cdot \nabla v) dx = -\int_U u \triangle v + \int_{\partial U} u \frac{\partial v}{\partial \nu} \, dS$,
- $\int_U (u \triangle v - v \triangle u) dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} dS - \int_{\partial U} v \frac{\partial u}{\partial \nu} dS$,

where n is the outward pointing unit vector for ∂U and dS is the area element for ∂U.

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Corollary

Let $U \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, and let $u \colon U \to \mathbb{R}$ be a harmonic function on U. Then

$$\int_{\partial U} \frac{\partial f}{\partial \nu} dS = 0,$$

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Lemma

The following functions are harmonic on $\mathbb{R}^n \setminus \{0\}$*:*

- $\Phi_2(x) = -\frac{1}{2\pi} \log |x|$, *for* $n = 2$,
- $\Phi_n(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}$, *for* $n > 2$,

where $\alpha(n)$ *is the volume of the unit ball in* $\mathbb{R}^n$.

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Theorem (Mean-value formula)

Let $u\colon U \to \mathbb{R}$ be a harmonic function on an open set U, let $x \in U$ and let $r > 0$ be such that $\overline{B_r(x)} \subset U$. Then

$$u(x) = \fint_{\partial B_r(x)} u dS.$$

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Proof.

Let

$$\varphi(r) = ⨍_{\partial B_r(x)} u(y)dS(y) = ⨍_{\partial B_1(0)} u(x+rz)dS(z)$$

Taking the derivative we obtain

$$\begin{aligned}\varphi'(r) &= ⨍_{\partial B_1(0)} Du(x+rz)z dS(z)\\ &= ⨍_{\partial B_1(0)} \frac{\partial u(x+rz)}{\partial \nu} dS(z)\\ &= c\int_{B_1(0)} \triangle u(x+rz)dz\\ &= 0.\end{aligned}$$

Hence φ is constant and $\lim_{r\to 0}\varphi(r) = u(x)$ □

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Exercise

Let $U \subset \mathbb{R}^n$ be a connected open set and let $u\colon U \to \mathbb{R}$ be a harmonic function. If u has a maximum at $x \in U$ then u constant.

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Definition

Given a bounded open set U with smooth boundary assume that for each $x \in U$ there is a function $\varphi^x \colon U \to \mathbb{R}$ such that

$$\begin{cases} \triangle \varphi^x & = 0 \\ \varphi^x(y) & = \Phi_n(x-y) \text{ on } \partial U. \end{cases}$$

The *Green function* for U is the function

$$G(x,y) = \Phi_n(x-y) - \varphi^x(y).$$

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Theorem

Given $u \in C^2(U)$ we have

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) - \int_U G(x,y)\triangle u(y)dy.$$

In particular, if u is harmonic,

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)dS(y).$$

Conversely, given $f\colon \partial U \to \mathbb{R}$ continuous, the expression

$$u(x) = -\int_{\partial U} f(y)\frac{\partial G}{\partial \nu}(x,y)dS(y).$$

defines a harmonic function u which equals f at ∂U.

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Proof.

Follows from several uses of Green's formulas.

$$\int_U (u\triangle v - v\triangle u)dx = \int_{\partial U} u\frac{\partial v}{\partial \nu}dS - \int_{\partial U} v\frac{\partial u}{\partial \nu}dS.$$

Fix x and consider $U\backslash B_r(x)$. Then, $\triangle_y G(x,y) = 0$ and $G(x,y) = 0$ on ∂U, hence

$$-\int_{U\backslash B_r(x)} G(x,y)\triangle u(y)dy = \int_{\partial U} u\frac{\partial G}{\partial \nu}dS - \int_{\partial B_r(x)} u\frac{\partial G}{\partial \nu} - G\frac{\partial u}{\partial \nu}dS$$

Need to estimate

$$\int_{\partial B_r(x)} u\frac{\partial G}{\partial \nu} \text{ and } -\int_{\partial B_r(x)} G\frac{\partial u}{\partial \nu}dS$$

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Proof.

$$\left| \int_{\partial B_r(x)} \Phi_n(x-y) \frac{\partial u}{\partial \nu} dS \right| \leq c \int_{\partial B_r(x)} r^{-n+2} \left| \frac{\partial u}{\partial \nu} \right| dS$$
$$\leq c r^{-n+2} r^{n-1} \xrightarrow{r \to 0} 0$$

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Proof.

$$\int_{\partial B_r(x)} u \frac{\partial \Phi_n(x-y)}{\partial \nu} dS = \fint_{\partial B_r(x)} u dS \xrightarrow{r \to 0} u(x)$$

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Theorem

Given a function $f \in C_c^2(\mathbb{R}^n)$*, let*

$$u(x) = \int_{\mathbb{R}^n} \Phi_n(x-y) f(y) dy.$$

Then $u \in C^2(\mathbb{R}^n)$ *and*

$$-\triangle u = f.$$