Analysis in one complex variable
Lecture 14 – Harmonic vs Holomorphic

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Recall

Lemma (Cauchy–Riemann relations)

If $f = u + iv$ is holomorphic, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \textit{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Corollary

If $f = u + iv$ is holomorphic, then u is harmonic.

Proof.

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x}.$$

□

Examples

Example

The function

$$u(x, y) = \log\left(\sqrt{x^2 + y^2}\right)$$

is harmonic because $u(z) = \operatorname{Re}\log(z)$.

Example

For any complex polynomial p, $\operatorname{Re}(p)$ is harmonic.

Example

The function $(x, y) \mapsto e^x \cos y$ is harmonic.

The operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$

- The operator $\frac{\partial}{\partial x}$ corresponds to directional derivative in the direction of e_1.
- The quantity dx is a covector for which $dx(e_1) = 1$ and $dx(e_j) = 0$ for $j \neq 1$.
- On $\mathbb{C}$ we can consider the complex covectors $dz = dx + idy$ and $d\bar{z} = dx - idy$.

  Notice that

$$dz(\frac{1}{2}(e_1 - ie_2)) = 1 \qquad dz(\frac{1}{2}(e_1 + ie_2)) = 0$$
$$d\bar{z}(\frac{1}{2}(e_1 - ie_2)) = 0 \qquad dz(\frac{1}{2}(e_1 + ie_2)) = 1.$$

- Introduce the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

The operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$

Lemma

A function $f : U \to \mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$, in which case

$$\frac{\partial f}{\partial z} = \frac{df}{dz}.$$

Lemma

$$4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} = \triangle.$$

Primitives

Lemma

Let $f = (u + iv)\colon U \to \mathbb{C}$ be holomorphic and consider the vector field $X = ue_1 - ve_2$. If $g = \varphi + i\psi$ is a holomorphic primitive for f, then $\nabla\varphi = X$

Proof.

Since g is holomorphic, we have, using the Cauchy–Riemann relations

$$(u + iv) = f = dg = \begin{pmatrix} \varphi_x & \varphi_y \\ -\varphi_y & \varphi_x \end{pmatrix} = (\varphi_x - i\varphi_y).$$

Therefore $\nabla\varphi = ue_1 - ve_2$. □

Harmonic $\Leftrightarrow$ holomorphic

Theorem

Let $U \subset \mathbb{C}$ be simply connected and let $u \colon U \to \mathbb{R}$ be a harmonic function. Then there is a holomorphic function $f \colon U \to \mathbb{C}$ whose real part is u.

Proof.

Insight: if there was such an f, by the Cauchy Riemann relations we would have

$$df = u_x - iu_y = 2\frac{\partial}{\partial z}u.$$

Consider the function $g = 2\frac{\partial}{\partial z}u$. Then

$$\frac{\partial g}{\partial \bar{z}} = 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}u = \frac{1}{2}\triangle u = 0.$$

Harmonic $\Leftrightarrow$ holomorphic

Proof.

So g is holomorphic, hence has a primitive, f. Let $u_1 = \mathrm{Re} f$, then

$$2\frac{\partial u_1}{\partial z} = \frac{\partial f}{\partial z} = g = 2\frac{\partial u}{\partial z}$$

Hence

$$\frac{\partial (u_1 - u)}{\partial z} = 0 \Rightarrow \frac{\partial (u_1 - u)}{\partial x} = 0 \text{ and } \frac{\partial (u_1 - u)}{\partial y} = 0$$

So $u_1 - u = c$ for some constant.
It follows that $\mathrm{Re}\,(f - c) = u$. $\square$